## Bond :-

A bond is an obligation by the bond issuer to pay money to the bond holder according to the rules specified at the time the bond is issued.

A bond pays a specific amount, its face value, or equivalently, its par value at the date of maturity. In addition, most bonds pay periodic coupon payments.

e.g. A 9% coupon bond with a face value of \$1000 will have a coupon of \$90 per year.

**Bid price** is the price dealers are willing to pay for the bond, and hence the prise at which the bond can be immediately.

**Ask price** :- It is price at ~~with~~ which dealers are willing to sell the bound, and hence the prize at which it can be bought immediately.

**Accrued interest** :- Suppose that a bond makes coupon payments every 6 months. If you purchase the bond midway through the coupon period, you will receive your first coupon payment only after 3 months. You are getting extra interest. So you must pay the first 3 months interest to the previous owner.

$$AI = \frac{\text{no. of days since last coupon}}{\text{no. of days in current coupon period}} \times \text{coupon amount}$$

Ex:- Suppose we purchase on May 8 a US Treasury bond that matures on August 15 in some distant year. The coupon rate is 9% . Coupon payments are made every feb 15 and Aug 15.

thus $AI = \frac{83}{83+99} \times 4.50 = 2.05$

This 2.05 would be added to the quoted price, expressed as a percentage of the face value, for example, $20.50 would be added to the bond if its face value were $1,000.

Quality Rating!-

Bonds that are either high or medium grade are considered to be investment grade. Bonds that are in or below the speculative category are often termed Junk bonds.

Yield!- It is the interest rate at which the present value of the stream of payment is exactly equal to the current price. This value is termed more properly the yield to maturity (YTM).

The YTM is just the internal rate of return of the bond at current price.

Suppose that a bond with face value $F$ makes $m$ coupon payments of $C/m$ each year and there are ~~so~~ $n$ periods remaining. The coupon payments sum to $C$ within a year. Suppose that the current price of the bond is $P$. Then the YTM is the value of $\lambda$ such that

$$P = \frac{F}{[1+(\lambda/m)]^n} + \sum_{k=1}^{n} \frac{C/m}{[1+(\lambda/m)]^k}$$

Bond price formula:- The price of a bond, having exactly $n$ coupon periods remaining to maturity and a YTM of $\lambda$, satisfy

$$P = \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left\{1 - \frac{1}{(1+(\lambda/m))^n}\right\}$$

where $F$ is the face value of bond, $C$ is the yearly coupon payment, and $m$ is no. of coupons payments per year.

Nature of Price - Yield Curve:-

Where the yield is exactly equal to the coupon rate, the bond is termed as Par - bond.

Price and yield have an inverse relation, ~~we~~ if yield goes up, price goes down.

As the maturity is increased, the price-yield curve becomes steeper.

Price of 9% coupon bonds

| | Yield | | | | |
|---|---|---|---|---|---|
| Time to maturity | 5% | 8% | 9% | 10% | 15% |
| 1 year | 103.85 | 100.94 | 100.00 | 99.07 | 94.61 |
| 5 years | 117.50 | 104.06 | 100.00 | 96.14 | 79.41 |
| 10 years | 131.18 | 106.80 | 100.00 | 93.77 | 69.42 |
| 20 years | 150.21 | 109.90 | 100.00 | 91.42 | 62.22 |
| 30 years | 161.82 | 111.31 | 100.00 | 90.54 | 60.52 |

## Current Yield.

$$CY = \frac{\text{annual coupon payment}}{\text{bond price}} \times 100$$

**Duration:-** The duration of a fixed-income instrument is a weighted average of the times that payments are made. The weighting coefficients are the present values of the individual cash flows.

Suppose that cash flows are received at $t_0, t_1, .., t_n$ times. Then duration of this stream is

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \dots + PV(t_n)t_n}{PV}$$

Where $PV(t_k)$ represents the present value of cash flow that occur at time $t_k$.
and PV is the present value of entire stream.

## Macaulay Duration :-

Suppose a financial instrument makes payment m times per year with the payment in period k being $c_k$, and there are n periods remaining. Then Macaulay duration D is

$$D = \frac{\sum_{k=1}^{n} (k/m) c_k / [1+(\lambda/m)]^k}{PV}$$

where $\lambda$ is the yield to maturity.

$$PV = \sum_{k=1}^{n} \frac{c_k}{[1+\lambda/m]^k}.$$

Ex:- Consider a 7% bond with 3 years to maturity. Assume that the bond is selling at 8% yield. Find the value of Macaulay duration.

a)

| A | B | C | D | E | F |
|---|---|---|---|---|---|
| Year | Payment | Discount factor @8% | Present Value of Payment (B×C) | Weight (D/Price) | (A×E) |
| 0.5 | 3.5 | 0.962 | 3.365 | 0.035 | 0.017 |
| 1 | 3.5 | 0.925 | 3.236 | 0.033 | 0.033 |

— see book. –

When the coupon payments are identical, then we have

<u>Macaulay duration formula</u>:- The Macaulay duration for a bond with a coupon rate $c$ per period, yield $y$ per period, $m$ periods per year, and exactly $n$ periods remaining, is

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}$$

<u>Ex</u>:- Consider 10%, 30 year bond. Let us assume that it is at par value, i.e. the yield is 10%.

then $c = y$.

$$\therefore \quad D = \frac{1+y}{my} - \frac{1+y+0}{my[(1+y)^n - 1] + my}$$

$$= \frac{1+y}{my}\left[1 - \frac{1}{(1+y)^n}\right]$$

Hence $D = \frac{1.05}{0.1}\left[1 - \frac{1}{(1.05)^{60}}\right] = 9.938.$

<u>Q1</u> An 8% bond with 18 years to maturity has a yield of 9%. What is the price of this bond? — 91.24.

## Qualitative Properties Of Duration

Duration of a coupon-paying bond is always less than its maturity.

Duration of a bond Yielding 5%.
as function of maturity & coupon rate

| | Coupon rate | | | |
|---|---|---|---|---|
| Years to Maturity | 1% | 2% | 5% | 10% |
| 1 | 0.997 | 0.995 | 0.988 | 0.977 |
| 2 | 1.984 | 1.969 | 1.928 | 1.868 |
| 5 | 4.875 | 4.763 | 4.485 | 4.156 |
| 10 | 9.416 | 8.950 | 7.989 | 7.107 |
| 25 | 20.164 | 17.715 | 14.536 | 12.754 |
| 100 | 22.572 | 21.200 | 20.363 | 20.067 |
| $\infty$ | 20.500 | 20.500 | 20.500 | 20.500 |

**Thm:-** Limiting value of duration as maturity is increased to infinity is

$$D \rightarrow \frac{1+\lambda/m}{\lambda}.$$

**Proof** By Macaulay duration formula.

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}.$$

Here let $y \to \lambda = my$,

then $D = \dfrac{1+\lambda/m}{\lambda} - \dfrac{1+y}{-}$

As $n \to \infty$

$$D \to \frac{1+\lambda/m}{\lambda}.$$

for example, $\lambda = 5\% = 0.05$, coupon rate $= 5\%$.
$m = 2$.

$$D \to \frac{1+\frac{0.05}{2}}{0.05} = 20.5.$$

## Duration and Sensitivity

In the case where payments are made $m$ times per year and yield is based on those same periods, we have

$$PV_k = \frac{c_k}{[1+(\lambda/m)]^k}$$

The derivative w.r.t. $\lambda$ is

$$\frac{d\,PV_k}{d\lambda} = \frac{-(k/m)\,c_k}{[1+(\lambda/m)]^{k+1}} = \frac{-k/m}{1+(\lambda/m)}\,PV_k.$$

We apply this expression for $P$.

$$P = \sum_{k=1}^{n} PV_k.$$

$$\frac{dP}{d\lambda} = \sum_{k=1}^{n} \frac{dPV_k}{d\lambda} = -\sum_{k=1}^{n} \frac{(k/m)PV_k}{1+\lambda/m} = -\frac{1}{1+(\lambda/m)} D.P$$

$$\equiv -D_m P$$

The value $D_M$ is called modified duration.

<u>Price Sensitivity formula</u>:- The derivative of price P with respect to the yield $\lambda$ of a fixed income security is

$$\frac{dP}{d\lambda} = -D_M P.$$

where $D_M = \frac{D}{1+\lambda/m}$ is the modified duration.

By using approximation, $\frac{dP}{d\lambda} \approx \frac{\Delta P}{\Delta\lambda}$.

we get $\Delta P \approx -D_M P \Delta\lambda$.

<u>Ex</u>:- 10% Coupon bond, Maturity = 30 years, then

$$D = 9.94$$

and $D_M = \frac{D}{1+\frac{0.1}{2}} = \frac{9.94}{1+0.05} = 9.47.$

Then if $\lambda$ is increased to 11% from 10%.

then $\Delta P = -D_m 100 \Delta\lambda = -947 \times 0.01 = -9.47.$

Hence $P \approx 90.53.$

Ex:- Consider 30 year zero-coupon bond. Suppose $\lambda = 10\%$. then $D = 30$ & $D_M \approx 27$.

if $\lambda$ changes to 11 from 10.

then $\Delta P = -D_M \, 100 \, \Delta\lambda$

$= -27 \times 100 \times 0.01 = -27$

$\Rightarrow \quad P \approx 73$

Q10 find the duration of a 10-year, 8% bond that is trading at a yield of 10%.

$\Rightarrow$ Here, $y = 0.05$, $m = 2$, $c = 0.04$, $n = 20$

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}$$

$$= \frac{1+0.05}{0.1} - \frac{1+0.05+20(0.04-0.05)}{2\times 0.04[(1+0.05)^{20}-1]+2\times 0.05}$$

$$= \frac{1.05}{0.1} - \frac{1.05 - 0.2}{0.08[1.65329] + 0.1}$$

$= 10.5 - 3.6596$

$\approx 6.84$