B.Sc (H) Mathematics
II Sem, SuA

Theory of Real functions
Section 4.2

Definition 4.2.1 Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c a cluster point of A. Then f is said to be bounded on a neighbourhood of c if $\exists$ a δ-neighbourhood $V_\delta(c)$ of c and a constant $M > 0$ $\ni$

We have $|f(x)| \leq M \quad \forall\, x \in A \cap V_\delta(c)$.

Thm 4.2.2 Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$. Then f ~~has~~ is bounded on some neighbourhood of c.

Pf:- Let $\lim_{x \to c} f(x) = L$ $\therefore$ for $\epsilon > 0$ $\exists\, \delta > 0$

$\ni 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Let $\epsilon = 1$ for this $\epsilon = 1$, $\exists\, \delta > 0 \ni$

$0 < |x - c| < \delta \Rightarrow |f(x) - L| < 1$

$|f(x)| - |L| \leq |f(x) - L| < 1$ whenever $0 < |x - c| < \delta$

$\Rightarrow |f(x)| < |L| + 1$, whenever $x \in (c - \delta, c + \delta)$, $x \neq c$

$\Rightarrow |f(x)| < |L| + 1$, provided $x \in V_\delta(c) \cap A$ (as $x \in A$)

$\Rightarrow$ $|f(x)| < |L| + 1$,

$\Rightarrow$

If $c \notin A$. then $|f(x)| \leq |L|+1$

Then bound M will be $|L|+1$.

If $c \in A$ then $M = \sup\{|f(c)|, |L|+1\}$

$\therefore$ In both cases, $|f(x)| \leq M \quad \forall x \in A \cap V_\delta(c)$.

$\Rightarrow$ f is bounded on a neighborhood of c.

Definition 4.2.3 Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$,

Sum $f+g : A \to \mathbb{R}$ defined as

$f+g(x) = f(x) + g(x)$

difference $f-g : A \to \mathbb{R}$ defined as

$f-g(x) = f(x) - g(x)$

Product $fg : A \to \mathbb{R}$ defined as $fg(x) = f(x)\,g(x)$

quotient $\frac{f}{g} : A \to \mathbb{R}$ defined as

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0 \quad \forall x \in A$.