B.Sc.(H) Mathematics — Sem IV, Sec B

Beta function

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In the last notes, we had discussed the improper integral of type I and type II. Before we start Beta function, let's look at some improper Integrals :-

A Useful Comparision Integral:-

Thm 1 The Improper Integral $\int_a^b \frac{dx}{(x-a)^n}$ Converges if and only if $n<1$.

Proof. If $n \le 0$ then $\int_a^b \frac{1}{(x-a)^n} dx$ will be proper. And if $n>0$ then $\int_a^b \frac{1}{(x-a)^n} dx$ will be improper as a being the point of discontinuity of the integrand.

Now for $n \neq 1$,

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{c \to a^+} \int_c^b \frac{1}{(x-a)^n}\,dx$$

$$= \lim_{c \to a^+} \left(\frac{(x-a)^{-n+1}}{1-n} \right)_c^b$$

$$= \lim_{c \to a^+} \left(\frac{(b-a)^{1-n}}{1-n} - \frac{(c-a)^{1-n}}{1-n} \right)$$

$$\cancel{= \begin{cases} \frac{(b-a)^{1-n}}{1-n} & \text{if } n\le 1 \\ \infty & \text{if } n>1 \end{cases}}$$

$c = a+h;\ h \to 0$

$$= \lim_{h \to 0} \left(\frac{(b-a)^{1-n}}{1-n} - \frac{h^{1-n}}{1-n} \right)$$

$$= \begin{cases} \frac{(b-a)^{1-n}}{1-n} & \text{if } n<1 \\ \infty & \text{if } n>1 \end{cases}$$

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ converges iff $n<1$.

Another Comparison Test:-

Thm 2 (Limit Comparision Test) Let $f(x)$ and $g(x)$ be two non-negative continuous function on $[a, b]$

Let $\lim_{x \to c} \frac{f(x)}{g(x)} = L$ with $L > 0$.

Then both $\int_c^b f(x)\,dx$ and $\int_c^b g(x)\,dx$ are

Convergent or divergent.

only STATEMENT WE HAVE TO REMEMBER

or

both $\int_a^c f(x)\,dx$ and $\int_a^c g(x)\,dx$ Converge or

diverge together.

Main Theorem Show that the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Converges $\Leftrightarrow$ m, n are +ve integers.

Proof. Case 1 Let $m \geq 1$, $n \geq 1$.

Then obviously $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is a proper one.

(For Ex:- if $m = 2$, $n = 3$ then $\int_0^1 x(1-x)^2\,dx = \frac{1}{12}$)

Case 2 Let $m < 1$, $n < 1$.

Take any no. b/w 0 & 1, say $1/2$

Then $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$

$+ \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx.$

Consider the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$,

let's denote this integral by I_1.

Since $m<1$, $I_1 = \int_0^{1/2} \frac{(1-x)^{n-1}}{x^{1-m}}\,dx$

Let $f(x) := \frac{(1-x)^{n-1}}{x^{1-m}}$, $g(x) = \frac{1}{x^{1-m}}$

Then $\frac{f(x)}{g(x)} = (1-x)^{n-1}$

Notice that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$. Therefore by Limit Comparison test $\int_0^{1/2} f(x)\,dx$ and $\int_0^{1/2} g(x)\,dx$ Converge or diverge together. But $\int_0^{1/2} g(x)$ $= \int_0^{1/2} \frac{1}{x^{1-m}}$ and it converges whenever $1-m<$ (by Thm 1) i.e. $\int_0^{1/2} g(x)\,dx$ Converges iff $m>0$.

and hence $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ Converges iff $m>0$.

Since $n<1$, denote the integral $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$

by I_2. i.e. $I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_{1/2}^{1} \frac{x^{m-1}}{(1-x)^{1-n}}\,dx$$

again, Mark $f(x)$ as $f(x) := \dfrac{x^{m-1}}{(1-x)^{1-n}}$

and $g(x) = \dfrac{1}{(1-x)^{1-n}}$, Notice that

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} x^{m-1} = 1$$

$\therefore$ By Limit Comparision Test, $\int_{1/2}^{1} \frac{x^{m-1}}{(1-x)^{1-n}}\,dx$

and $\int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}$ Converge or diverge together.

But $\int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}$ Converges iff $1-n<1$, by Thm 1.

i.e. $\int_{1/2}^{1} g(x)$ Converges iff $n > 0$.

Thus I_2 Converges iff $n > 0$

and Hence $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ Converges iff $m>0$, $n>0$.

This Integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, for $m, n > 0$, is called the Beta function & denoted by $\beta(m, n)$.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

In Particular, $\beta(2,3) = \int_0^1 x(1-x)^2 dx = \frac{1}{12}$

and so on.