Section 2.4

In the earlier sections we have studied linear transformation and how a matrix is associated with a linear transformation. In this section we see that a linear transformation is associated with every a matrix and study isomorphisms between vector spaces.

Defn:- Let V and W be vector spaces and $T: V \to W$ be linear. A function $U: W \to V$ is said to be inverse of T if $TU = I_W$ and $UT = I_V$. If T has inverse, then T is said to be invertible and its inverse is denoted by T^{-1}.

Properties of invertible functions:-

(1.) Let $T: V \to W$ and $U: W \to V$ be invertible functions, then TU is invertible and $(TU)^{-1} = U^{-1}T^{-1}$.

Proof:- As, $TU: W \to W$

and $(TU)(U^{-1}T^{-1}) = T(UU^{-1})T^{-1}$

$= T(I)T^{-1}$

$= (TI_V)T^{-1}$

$= TT^{-1} = I_W$

also $(U^{-1}T^{-1})TU = U^{-1}(T^{-1}T)U$

$= U^{-1}I_W U = U^{-1}U = I_W$

$\therefore$ TU is invertible and $(TU)^{-1} = U^{-1}T^{-1}$.

(2.) If $T: V \to W$ is invertible, then T^{-1} is also invertible and $(T^{-1})^{-1} = T$.

Proof:- As T is invertible.

$\therefore$ $TT^{-1} = I_W$ $\quad T^{-1}T = I_V$

$\Rightarrow$ T^{-1} is also invertible

and $(T^{-1})^{-1} = T$.

(3.) ~~If~~ $T: V \to W$ is invertible if and only if T is one-one and onto.

Proof:- Do yourself.

(4.) Let $T: V \to W$ be linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if $\text{rank}(T) = \dim(V)$

Proof:- If T is invertible, then T is one-one and onto.

$\therefore$ By theorem 2.5, $\dim(V) = \text{ran}(T)$.

Conversely If $\text{rank}(T) = \dim(V)$

Then By theorem 2.5, T is one-one and onto.

$\Rightarrow$ T is invertible.

Examples:-

(1.) let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(a+bx) = (a, a+b)$$

As $\dim(P_1(\mathbb{R})) = \dim(\mathbb{R}^2) = 2$. If we show that T is one-one, then T is invertible [By thm 2.5 & property].

let $a_1+b_1x,\ a_2+b_2x \in P_1(\mathbb{R})$ s.t.

$$T(a_1+b_1x) = T(a_2+b_2x)$$

$$(a_1, a_1+b_1) = (a_2, a_2+b_2)$$

$\Rightarrow\ a_1 = a_2$ and $a_1+b_1 = a_2+b_2$

$\Rightarrow\ a_1 = a_2$ and $b_1 = b_2$

$\Rightarrow\ a_1+b_1x = a_2+b_2x$.

$\therefore$ T is one-one.

Hence T is invertible

and $T^{-1}: \mathbb{R}^2 \to P_1(\mathbb{R})$ is defined as.

$$T^{-1}(a,b) = a+(b-a)x.$$

(2.) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$$

Check whether T is invertible or not.

Sol By dimension theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^2) = 2.$$

$$\Rightarrow\ \text{rank}(T) \leq 2.$$

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$\Rightarrow$ rank$(T) \neq \dim(\mathbb{R}^3) = 3$.

$\Rightarrow$ $R(T) \neq \mathbb{R}^3$

$\therefore$ T is not onto.

$\Rightarrow$ T is not invertible.

(3.) Let $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

Check whether T is invertible or not.

Solⁿ Define $T^{-1}: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ as

$$T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix}$$

then $TT^{-1} = T^{-1}T = I$ (check).

$\therefore$ T is invertible.

Thm 2.17 Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Then $T^{-1}: W \to V$ is linear

Proof:- Let $y_1, y_2 \in W$ and $c \in F$

As T is invertible $\Rightarrow$ T is one-one and onto.

$\therefore$ there exists unique $x_1, x_2 \in V$ s.t. $T(x_1) = y_1$ & $T(x_2) = y_2$

$\Rightarrow x_1 = T^{-1}(y_1)$ & $x_2 = T^{-1}(y_2)$.

So,

$$\begin{aligned} T^{-1}(cy_1 + y_2) &= T^{-1}(cT(x_1) + T(x_2)) \\ &= T^{-1}(T(cx_1 + x_2)) \\ &= cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2) \end{aligned}$$

Def^n:- Let A be an $n \times n$ matrix. Then A is invertible if exists an $n \times n$ matrix B such that

$AB = BA = I$.

① Lemma:- Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case, $\dim(V) = \dim(W)$.

Proof:- Suppose V is finite-dimensional and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V.

then $R(T) = \text{Span}(T(\beta))$ (By theorem 2.2).

and $R(T) = W$ [$\because T$ is onto)

$\therefore$ W is finite-dimensional.

Conversely Let W is finite-dimensional.

then $T^{-1}: W \to V$ is linear & invertible.

Applying above argument we get V is finite-dim.

Now suppose that V and W are both finite-dimensional.

Claim:- $\dim(V) = \dim(W)$

As T being invertible, T is one-one ~~[illegible]~~

$\Rightarrow$ nullity$(T) = 0$

and T is onto $\Rightarrow$ ~~$R(T) = W$~~ $R(T) = W$.

$\Rightarrow \dim(R(T)) = \dim(W)$

and By dimension theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

$$\rightarrow \text{rank}(T) = \dim V$$

$$\rightarrow \dim W = \dim V$$

Thm 2.18 Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, let $T: V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Proof:- Let T be invertible.

$\therefore$ By lemma, $\dim(V) = \dim(W)$

and let $\dim(V) = n$. Therefore $[T]_\beta^\gamma$ is an $n \times n$ matrix

and $TT^{-1} = I_W$ and $T^{-1}T = I_V$.

Thus,

$$I_n = [I_V]_\beta = [T^{-1}T]_\beta$$

$$= [T^{-1}]_\gamma^\beta [T]_\beta^\gamma \quad \text{[By thm 2.11]}$$

Also,

$$I_n = [I_W]_\gamma = [TT^{-1}]_\gamma$$

$$= [T]_\beta^\gamma [T^{-1}]_\gamma^\beta \quad \text{[By thm 2.11]}$$

$$\rightarrow I_n = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

$\therefore$ $[T]_\beta^\gamma$ is invertible and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

Conversely. Suppose $A = [T]_\beta^\gamma$ is invertible.

Then there exists a $n \times n$ matrix such that

$$AB = BA = I_n.$$

Claim:- T is invertible.

Define $U: W \to V$ as

$$U(w_j) = \sum_{i=1}^{n} B_{ij} v_i \quad \text{for } j = 1, 2, \to n$$

where $\gamma = \{w_1, w_2, -, w_n\}$ and $\beta = \{v_1, v_2, -, v_n\}$.

Then U is linear (show it).

and $[U]_\gamma^\beta = B$

Now we show that U is inverse of T.

As $$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta.$$

$$\Rightarrow \quad UT = I_V \quad -(1)$$

also $$[TU]_\gamma = [T]_\beta^\gamma [U]_\gamma^\beta = AB = I_n = [I_W]_\gamma$$

$$\Rightarrow \quad TU = I_W. \quad -(2)$$

from (1) and (2), T is invertible and $T^{-1} = U$. ∎

Corollary:- Let V be a finite-dimensional vector space with an ordered basis β, and let $T: V \to V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

Proof:- Take $W = V$ in thm 2.18 ■.

Corollary 2:- Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. furthermore, $(L_A)^{-1} = L_{A^{-1}}$.

Proof:- Recall $L_A : F^n \to F^n$ defined as

$$L_A(x) = Ax.$$

and By thm 2.15 (a) $[L_A]_\beta = A$ where β is standard ordered bases for F^n.

Now consider A is invertible, then $[L_A]_\beta$ is invertible, then By corollary ①, L_A is invertible.

conversely If L_A is invertible.

Then $[L_A]_\beta$ is invertible [By corollary ①]

$\therefore$ A is invertible.

Claim:- $(L_A)^{-1} = L_{A^{-1}}$.

Consider

$$\begin{aligned} L_A L_{A^{-1}} &= L_{AA^{-1}} && (\text{By thm } 2.15) \\ &= L_{I_n} && (\because \text{As } AA^{-1} = I_n) \\ &= I_{F^n} && (\text{By thm } 2.15) \end{aligned}$$

Similarly $L_{A^{-1}} L_A = I_{F^n}$

$\therefore$ $(L_A)^{-1} = L_{A^{-1}}$.

Now we'll define the isomorphism between two vector spaces.

<u>Def^n</u>:- Let V and W be vector spaces. We say that V is <u>isomorphic</u> to W if there exists a linear transformation $T: V \to W$ that is invertible. Such a linear transformation is called <u>isomorphism</u> from V onto W.

<u>Examples</u>:-

(1.) Define $T: F^2 \to P_1(F)$ by

$$T(a_1, a_2) = a_1 + a_2 x.$$

then T is invertible and linear. (Do it).

$\therefore$ T is isomorphism

$\therefore$ F^2 is isomorphic to $P_1(F)$.

(2.) Define $T: F^4 \to M_{2\times 2}(F)$ as

$$T(a_1, a_2, a_3, a_4) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

then T is linear and invertible (Do it)

$\therefore$ T is isomorphism.

$\Rightarrow$ F^4 is isomorphic to $M_{2\times 2}(F)$.

(3.) Define $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ as.

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{pmatrix} a_0 & a_2 \\ a_3 & a_1 \end{pmatrix}$$

then T is linear and invertible (Show it)

$\therefore$ T is an isomorphism.

$\Rightarrow$ $P_3(R)$ is isomorphic to $M_{2\times 2}(R)$.

<u>Thm 2.19</u> Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

<u>Proof</u>:- Let V is isomorphic to W.

$\Rightarrow$ there exists $T: V \to W$ such that T is invertible linear transformation.

$\therefore$ By ~~By~~ lemma (1), $\dim(V) = \dim(W)$.

<u>Conversely</u>, let $\dim(V) = \dim(W) = n$ and let $\beta = \{v_1, v_2, -, v_n\}$ and $\gamma = \{w_1, w_2, -, w_n\}$ be ordered bases for V and W respectively.

Then By theorem 2.6, there exists a linear transformation $T: V \to W$ such that

$$T(v_i) = w_i \quad \text{for all } i = 1, 2, -, n.$$

and
$$\begin{aligned} R(T) &= \text{span}(T(\beta)) && (\text{By thm. } 2.2) \\ &= \text{Span}(\gamma) && (\because T(\beta) = \gamma) \\ &= W. \end{aligned}$$

$\therefore$ $R(T) = W$

$\Rightarrow$ T is onto.

$\therefore$ By thm 2.5, T is one-one, Hence T is isomorphism.

Corollary:- Let V be a vector space F. Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Proof:- Suppose V is isomorphic to F^n.

Then $\dim(V) = \dim(F^n)$ ($\because$ By thm 2.19)

But $\dim(F^n) = n$

$\therefore \dim(V) = n$.

Conversely, let $\dim(V) = n$

Also, $\dim(F^n) = n$

$\therefore \dim(V) = \dim(F^n)$

$\Rightarrow$ V is isomorphic to F^n (By thm 2.19).

Thm 2.20:- Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the function $\phi : \mathcal{L}(V,W) \rightarrow M_{m\times n}(F)$, defined by

$$\phi(T) = [T]_\beta^\gamma \quad \text{for } T \in \mathcal{L}(V,W),$$

is an isomorphism.

Proof:- To show ϕ an isomorphism, we need to show that ϕ is linear, one-one and onto.

Linear, ϕ is linear, from theorem 2.8.

one-one, let $T, U \in \mathcal{L}(V,W)$ s.t.

$$\phi(T) = \phi(U)$$

$\Rightarrow [T]_\beta^\gamma = [U]_\beta^\gamma$

$\Rightarrow T = U$ $\quad$ ($\because$ If $[T]_\beta^\gamma = [U]_\beta^\gamma \Rightarrow T = U$).

$\therefore$ ϕ is one-one.

onto: Let $A \in M_{m\times n}(F)$, A is $m \times n$ matrix.

Define $T: V \rightarrow W$ as

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i \quad \text{for } 1 \le j \le n$$

where $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ be ordered bases of V and W respectively.

Then T is linear [show it].

and $[T]_\beta^\gamma = A$.

or $\phi(T) = A$

$\therefore$ ϕ is onto.

Therefore ϕ is an isomorphism.

Corollary:- Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then $\mathcal{L}(V, W)$ is finite-dimensional of dimension mn.

Proof:- As $\mathcal{L}(V,W)$ is isomorphic to $M_{m\times n}(F)$

$\therefore$ $\dim(\mathcal{L}(V,W)) = \dim(M_{m\times n}(F)) = mn$

Q17 (Page 108)

<u>Corollary</u>:- Let V and W be finite-dimensional vector spaces and $T: V \to W$ be an isomorphism. Let V_0 be a subspace of V. Then

(a) $T(V_0)$ is a subspace of W.

(b) and $\dim(V_0) = \dim(T(V_0))$.

<u>Proof</u>:- (a) $T(V_0) = \{T(x) : x \in V_0\}$.

As V_0 is subspace of V.

$\therefore \; 0_V \in V_0$

$\Rightarrow T(0_V) \in T(V_0)$

$\Rightarrow \quad 0_W \in T(V_0) \qquad (\because T(0) = 0)$.

Now let $x, y \in T(V_0)$, then there exists $x', y' \in V_0$ such that $T(x') = x$ & $T(y') = y$.

Then $x + y = T(x') + T(y')$

$= T(x' + y')$

$\Rightarrow x + y \in T(V_0)$ as $x' + y' \in V_0$.

Let $\alpha \in F$.

then $\alpha x = \alpha T(x')$

$= T(\alpha x')$

$\therefore \; \alpha x \in T(V_0)$ as $\alpha x' \in V_0$.

$\Rightarrow T(V_0)$ is subspace of W.

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(b) As $T: V \to W$ is an isomorphism.

$T|_{V_0} \to T(V_0)$ is again an isomorphism (show it)

restriction of T to V_0

$\Rightarrow$ V_0 is isomorphic to $T(V_0)$

$\therefore$ $\dim(V_0) = \dim(T(V_0))$

Def^n:- Let β be an ordered basis for an n-dimensional vector space V ~~over~~ over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$.

Thm 2.21 for any finite-dimensional vector space V with ordered bases β, ϕ_β is an isomorphism.

Proof:- Let $\dim(V) = n$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for V.

Then $\phi_\beta : V \to F^n$ defined as

$$\phi_\beta(x) = [x]_\beta$$

Claim:- ϕ_β is isomorphism, ~~for that it is sufficient to show that ϕ_β is linear~~.

first we show that ϕ_β is linear.

Let $c \in F$ and $x, y \in V$

$\therefore \exists\ \alpha_1, \alpha_2, \ldots, \alpha_n \in F$ and $a_1, a_2, \ldots, a_n \in F$ s.t.

$x = \sum_{i=1}^{n} \alpha_i v_i$ and $y = \sum_{i=1}^{n} a_i v_i$.

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then $[x]_\beta = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and $[y]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

and $cx+y = (c\alpha_1 + a_1)v_1 + \cdots + (c\alpha_n + a_n)v_n$

$\therefore [cx+y]_\beta = \begin{pmatrix} c\alpha_1 + a_1 \\ \vdots \\ c\alpha_n + a_n \end{pmatrix} = c\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$= c[x]_\beta + [y]_\beta$.

$\therefore \phi_\beta(cx+y) = [cx+y]_\beta = c[x]_\beta + [y]_\beta$

$= c\,\phi_\beta(x) + \phi_\beta(y)$

$\therefore \phi_\beta$ is linear.

Now to show ϕ_β is isomorphism, it is enough to show that ϕ_β is one-one [$\because$ Thm 2.5]

Let $x, y \in V$ s.t. $\phi_\beta(x) = \phi_\beta(y)$

$\Rightarrow [x]_\beta = [y]_\beta$

$\Rightarrow x = y$ (show it) (?)

$\therefore \phi_\beta$ is one-one

$\therefore \phi_\beta$ is an isomorphism.

Thm:- (Relationship between linear transformation $T: V \to W$ and $L_A: F^n \to F^m$).

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Let V and W be vector spaces of dimensions n and m, respectively and let $T: V \to W$ be a linear transformation, let β and γ are arbitrary ordered bases of V and W respectively, and let $A = [T]_\beta^\gamma$, then following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] \arrow[dr, phantom, "\curvearrowright"] & W \arrow[d, "\phi_\gamma"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

i.e. $L_A \phi_\beta = \phi_\gamma T$.

Proof:- Let $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$ be the ordered bases of V and W respectively.

and $L_A \phi_\beta : V \to F^m$ and $\phi_\gamma T : V \to F^m$

To show $L_A \phi_\beta = \phi_\gamma T$ it is sufficient to show that $L_A \phi_\beta(v_i) = \phi_\gamma T(v_i) \quad \forall i = 1, 2, \dots, n$.

Consider $L_A \phi_\beta(v_i) = L_A [v_i]_\beta$

$= A[v_i]_\beta$

But $[v_i]_\beta = (0, 0, \dots, 0, \underset{\downarrow}{1}, 0, \dots 0)^T$ or $[v_i]_\beta = e_i$

1 is at ith place.

$\therefore \quad A[v_i]_\beta = i$th column of A.

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$\therefore$ $L_A \phi_\beta(v_i)$ is i^{th} column of A.

And $\phi_\gamma T(v_i) = \phi_\gamma\left(\sum_{j=1}^{m} A_{ji} w_j\right)$

$= \left[\sum_{j=1}^{m} A_{ji} w_j\right]_\gamma$

$=$ i^{th} column of A.

$\therefore$ $L_A \phi_\beta(v_i) = \phi_\gamma T(v_i)$ for $\forall\, i = 1, 2, -, n$.

$\Rightarrow$ $L_A \phi_\beta = \phi_\gamma T$.

Qn:- Let $\sim$ mean "is isomorphic to". Prove that $\sim$ is an equivalence relation on the class of finite-dimensional vector spaces over F.

Soln:- Let V be a vector space over F.

then $\dim(V) = \dim(V)$

$\therefore$ V is isomorphic to V.

$\Rightarrow$ $V \sim V$

$\Rightarrow$ $\sim$ is reflexive.

Symmetric Let $V \sim W$.

$\Rightarrow$ V is isomorphic to W.

$\Rightarrow$ $\exists$ $T: V \to W$ an isomorphism.

then $T^{-1}: W \to V$ is an isomorphism.

$\Rightarrow$ $W \sim V$

$\therefore$ $\sim$ is symmetric.

transitive: Let V, W and Z be finite dimensional vector spaces over F s.t.

$$V \sim W \quad \text{and} \quad W \sim Z$$

As $V \sim W \Rightarrow \dim(V) = \dim(W)$

and $W \sim Z \Rightarrow \dim(W) = \dim(Z)$

$$\therefore \dim(V) = \dim(Z)$$

$$\Rightarrow V \sim Z$$

$\therefore \sim$ is transitive.

$\therefore \sim$ is an equivalence relation.

Section 2.5

In the previous sections we have studied that a matrix is associated with a linear transformation on finite-dimensional vector spaces and seen how a coordinate vector relative to a basis of a vector space. Now suppose vector space V have two basis β and β', A question arises: How can a coordinate vector relative to one basis be changed into a coordinate vector relative to other? To answer this question, we have following result.

<u>Thm 2.22</u> Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$, Then

(a) Q is invertible

(b) for any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

[The Q in this theorem is called <u>change of coordinate matrix</u> and Q changes β'-coordinate into β-coordinate.

<u>Proof:</u> (a) As $I_V : V \to V$ is defined as

$$I_V(x) = x$$

then clearly I_V is invertible.

$\therefore$ $[I_V]_{\beta'}^{\beta} = Q$ is invertible [By thm 2.18].

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(b) for any $v \in V$

$$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta} [v]_{\beta'} \quad [\text{By thm } 2.14]$$

$$= Q [v]_{\beta'} \quad \blacksquare$$

Examples:-

(1.) In $\mathbb{R}^2$, let $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$. find Q.

Sol^n As $Q = [I_V]_{\beta'}^{\beta}$

$$I_V(a_1, a_2) = (a_1, a_2) = a_1 . e_1 + a_2 . e_2$$

and $I_V(b_1, b_2) = (b_1, b_2) = b_1 . e_1 + b_2 . e_2$.

$$\therefore \quad Q = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

(2.) In $\mathbb{R}^2$, let $\beta = \{(2,5), (-1,-3)\}$ & $\beta' = \{e_1, e_2\}$. find Q.

Sol^n $I_V(e_1) = e_1 = (1, 0)$

$$= 3 . (2,5) + 5 . (-1,-3)$$

and $I_V(e_2) = e_2 = (0, 1)$

$$= -1 . (2,5) + (-2)(-1,-3)$$

$$\therefore \quad Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}.$$

③ In $P_2(\mathbb{R})$, let $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and
$$\beta' = \{1, x, x^2\}$$
then find $Q = [I_V]_{\beta'}^{\beta}$.

Solⁿ: As $\quad 1 = 0.(2x^2 - x) + 1.(3x^2 + 1) + (-3).x^2$

and $\quad x = -1(2x^2 - x) + 0.(3x^2 + 1) + 2.x^2$

and $\quad x^2 = 0.(2x^2 - x) + 0.(3x^2 + 1) + 1.x^2$

$$\therefore \quad Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}.$$

Note:- A linear transformation that maps a vector space V into itself is called linear operator.

Thm 2.23 :- Let T be a linear operator on a finite dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinate into β-coordinate. Then
$$[T]_{\beta'} = Q^{-1} [T]_{\beta} Q.$$

Proof:- Let I be the identity transformation on V. then $\quad TI = IT = T$.

Then

$$Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'} \qquad [\because Q = [I]_{\beta'}^{\beta}]$$

$$= [IT]_{\beta'}^{\beta} \qquad [\because \text{Thm } 2.11]$$

$$= [TI]_{\beta'}^{\beta}$$

$$= [T]_{\beta}[I]_{\beta'}^{\beta} = [T]_{\beta} Q$$

$$\therefore \quad Q[T]_{\beta'} = [T]_{\beta} Q$$

~~Post~~ Pre multiplying by Q^{-1}, we get

$$[T]_{\beta'} = Q^{-1}[T]_{\beta} Q. \quad \blacksquare$$

Examples:-

(1) Let T be the linear operator on $\mathbb{R}^2$ defined by

Q.4 (Page 116)

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let $\beta = \{e_1, e_2\}$ and $\beta' = \{(1,1),(1,2)\}$ be ordered bases for $\mathbb{R}^2$. Find $[T]_{\beta'}$ $\left[\text{Use } \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\right]$

Sol^n: ~~first~~ first we find Q.

$$(1,1) = 1.e_1 + 1.e_2$$

and $$(1,2) = 1.e_1 + 2.e_2$$

$$\therefore \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

and $T(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $T(e_2) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

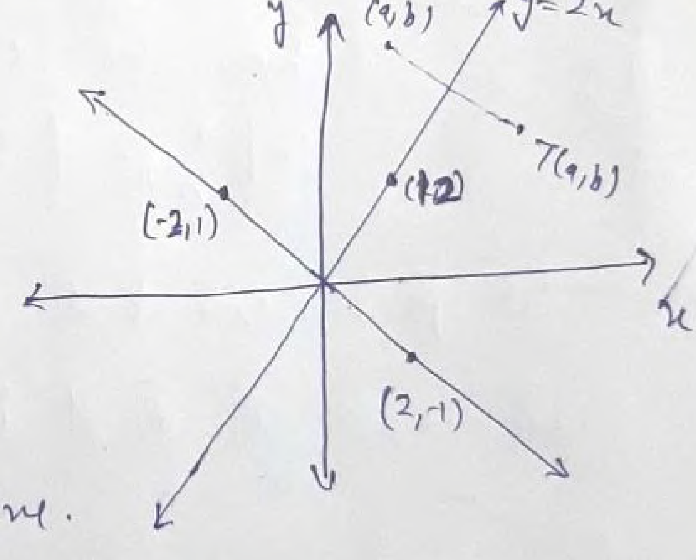
$\therefore \quad [T]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$

$$\therefore \quad [T]_{\beta'} = Q^{-1} [T]_\beta \, Q$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}.$$

2. Let T be a reflection about the line $y = 2x$. find expression for $T(a,b)$ for any (a,b) in $\mathbb{R}^2$.



Solⁿ: Here $T(a,b)$ is the reflection of (a,b) about the line $y = 2x$.

Clearly $T(1,2) = (1,2)$

As $(1,2)$ lies on $y = 2x$ line.

and $T(-2,1) = -(-2,1) = (2,-1)$

Therefore we let $\beta' = \{(1,2), (-2,1)\}$

then β' is an ordered basis for $\mathbb{R}^2$ (check)

Let $\beta = \{e_1, e_2\}$ be the standard ordered basis for $\mathbb{R}^2$.

then $Q = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

and $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So if ~~of~~ we find $[T]_\beta$, we can find T.

As $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$\Rightarrow$ $[T]_\beta = Q[T]_{\beta'}Q^{-1}$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix}$$

$$[T]_\beta = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

Then T is left multiplication by $[T]_\beta$.

$\therefore$ ~~$T(a,b) = \frac{1}{5}$~~

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3a+4b \\ 4a+3b \end{pmatrix}.$$

Corollary:- Let $A \in M_{n\times n}(F)$, and let γ be an ordered bases for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n\times n$ matrix whose j^{th} column is the j^{th} vector of γ.

Proof:- Let β be the standard ordered bases for F^n.

Then $[L_A]_\beta = A$ (thm 2.15)

and let $Q = [I]_\gamma^\beta$ then j^{th} column of Q is the j^{th} vector of γ.

then $\quad [L_A]_\gamma = Q^{-1}[L_A]_\beta Q$
$$= Q^{-1} A Q \quad \blacksquare$$

<u>Examples:-</u>

1. Let $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

then find Q and $[L_A]_\gamma$ such that $[L_A]_\gamma = Q^{-1} A Q$.

<u>Sol</u>:- By corollary, j^{th} column of Q is j^{th} column vector of γ.

$$\therefore \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\quad Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$\therefore \quad [L_A]_\gamma = Q^{-1} A Q$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ and $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

find $[L_A]_\gamma$.

Sol^n:- Here $Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

then $Q^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$$\therefore \quad [L_A]_\gamma = Q^{-1} A Q$$

$$= \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}.$$

Def^n:- Let A and B be matrices in $M_{n\times n}(F)$. We say that B is <u>similar</u> to A if there exists an invertible matrix Q s.t. $B = Q^{-1} A Q$.

Note!- Relation of similarity is an equivalence relation (Do it).

Using the above def^n, thm 2.23 can be stated as: If T is a linear operator on a finite-dimensional ~~vector~~ vector-space V, and if β and β' are any ordered bases for V, then $[T]_\beta$ is similar to $[T]_{\beta'}$. ∎