Linear Combination:

Let V be a vector space over a field F. A vector $v \in V$ is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_n$, if there exist scalars $a_1, a_2, \ldots, a_n$ in F such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

For example

i) Let $V = \mathbb{R}^2$

and let $v = (8,4)$, $v_1 = (1,2)$, $v_2 = (2,0)$

then

$$v = 2(1,2) + 3(2,0)$$

i.e

$$\boxed{v = a_1 v_1 + a_2 v_2}$$

where $a_1 = 2$, $a_2 = 3$.

(ii) Consider, $V = \mathbb{R}^3$.

Let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$

$v_3 = (0,0,1)$ and $v = (2,5,7)$

Then

$$(2,5,7) = 2(1,0,0) + 5(0,1,0) + 7(0,0,1)$$

i.e

$$\boxed{v = a_1 v_1 + a_2 v_2 + a_3 v_3}$$

where $a_1 = 2$, $a_2 = 5$, $a_3 = 7$

In general,

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

Thus, every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$ and $v_3 = (0,0,1)$.

OR

In other words we can say the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ spans (or generates) $\mathbb{R}^3$.

Que: Express the vector $(2,5,10)$ as a linear combination of the vectors $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$.

Sol^n Let $(a,b,c) \in \mathbb{R}^3$ and x, y, z are real number (scalars) such that

$$(a,b,c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$$

$$\therefore \quad x+y+z = a$$
$$x+y = b$$
$$x = c$$

solving these equations

$$x = c, \quad y = b-c, \quad z = a-b$$

$$\therefore (a,b,c) = c(1,1,1) + (b-c)(1,1,0) + (a-b)(1,0,0)$$

Thus $(2,5,10) = 10(1,1,1) + (5-10)(1,1,0) + (2-5)(1,0,0)$

$= 10(1,1,1) - 5(1,1,0) - 3(1,0,0)$

Here $x = 10$, $y = -5$, $z = -3$.