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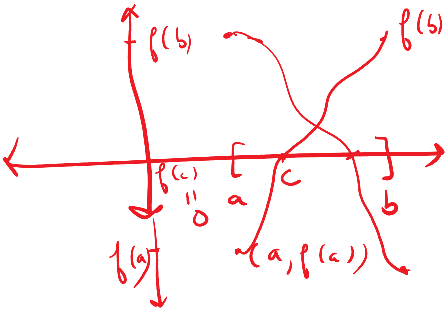
13:55

$f(x) = 0$

Location of roots theorem (only statement)

"Bisection Method"

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or if $f(b) < 0 < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = 0$



$f(a) < 0 < f(b)$
$\downarrow$
$f(c)$

Q1. Show that the given equations have roots in the specified intervals

(1) $x e^x - 2 = 0$ in $[0, 1]$

Let $f(x) = x e^x - 2 \quad \forall\, x \in [0, 1]$

$f(0) = -2$ & $f(1) = e - 2 > 0 \quad (e > 2)$

$\therefore f(0) < 0 < f(1)$

Also f is cts on $[0, 1]$

$x \to x$ is cts (identity)
$x \to e^x$ is cts
$\Rightarrow x \to x e^x$ is cts (prod of cts)
Also $x \to -2$ is cts (const)
$\therefore x \to x e^x - 2$ is cts (sum)

By location of roots theorem

$\exists\, c \in (0, 1) : f(c) = 0$

$f(c) = 0$
$c e^c - 2 = 0$
$\boxed{c e^c = 2}$
$c \in (0, 1)$

$c e^c = 2$

(2) $x = \cos x$ in $[0, \pi/2]$

Let $f(x) = \cos x - x, \ \forall\, x \in [0, \pi/2]$

f is cts on $[0, \pi/2]$ (REASON)

$f(0) = 1$ & $f(\pi/2) = -\pi/2$

$\therefore f(0) > 0 > f(\pi/2)$

$\therefore$ By location roots theorem $\exists\, c \in (0, \pi/2) : f(c) = 0$

$\Rightarrow \cos c = c, \ c \in (0, \pi/2)$

Q2. Let $I = [a, b]$ & $f : I \to \mathbb{R}$ & $g : I \to \mathbb{R}$ be cts on I. Show that if the set $E = \{x \in I \mid f(x) = g(x)\}$ has the property that $(x_n) \subseteq E$ & $x_n \to x_0$ then show that x of E

Sol

Given f, g cts ; $x_n \in E$, $x_n \to x_0$

To show $x_0 \in E$ (i.e $f(x_0) = g(x_0)$)

$x_n \to x_0$, f & g are cts at $x_0 \in I$ $\therefore$ by seq criterion

as $x_n \in I$ $\forall n$ & $x_n \to x_0$ $\Rightarrow x_0 \in I$

$f(x_n) \to f(x_0)$ & $g(x_n) \to g(x_0)$

As $x_n \in E \Rightarrow f(x_n) = g(x_n)$ $\forall n$

$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)$

$\Rightarrow f(x_0) = g(x_0)$

$\Rightarrow x_0 \in E$

($x_0 \in I$ & $f(x_0) = g(x_0)$

$a \le x_n \le b$

$\Rightarrow a \le \lim x_n \le b$)

$\lim x_n = x_0$

Q4 Show that every poly of odd degree with real coeff has at least one real root.

Sol Let $p(x) = a_n x^{n} + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ ($a_n \neq 0$, n is odd, $a_k \in \mathbb{R}$)

$x^3 + 2x^2 + 3x + 1$

$= x^n \left[a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right]$

If $a_n > 0$

$\lim_{x\to\infty} p(x) = \infty \; (>0)$

$\boxed{p(\infty)}$

$\lim_{x\to-\infty} p(x) = -\infty \; (<0)$

$(-\infty, \infty)$

n odd

$\lim_{x\to\infty} a_n x^n = \infty$

$\lim_{x\to-\infty} a_n x^n = -\infty$

$\lim_{x\to\infty} \frac{1}{x^k} = 0$

$\lim_{x\to-\infty} p(x) < 0 < \lim_{x\to\infty} p(x)$

$p(b) < 0 < p(a)$

$p(-\infty)$ $\qquad$ $p(\infty)$

Also, p is cts on $\mathbb{R}$ (poly function)

$\therefore$ by location of roots thm $\exists \; c \in (-\infty, \infty)$

i.e $c \in \mathbb{R}$

such that $p(c) = 0$

$\therefore$ $p(x)$ has a real root

the odd degree poly

x deg $\therefore$ $p(x)$ has a real root

The case $a_n < 0$ is similar.