External Direct Products

(Q-1) Show that $G \oplus H$ is Abelian iff G & H are Abelian

(Solution) Let $G \oplus H$ is Abelian

to show: G & H are Abelian

Let $g_1, g_2 \in G$ & $h_1, h_2 \in H$

Now, consider $(g_1,h_1)(g_2,h_2) = (g_2,h_2)(g_1,h_1)$ (by using $G \oplus H$ is abelian)

$\Rightarrow (g_1 g_2, h_1 h_2) = (g_2 g_1, h_2 h_1)$

$\Rightarrow g_1 g_2 = g_2 g_1$ & $h_1 h_2 = h_2 h_1$

$\therefore$ G & H are both abelian groups

Conversely, Let $(g_1,h_1), (g_2,h_2) \in G \oplus H$

& G & H are abelian

Now, $(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 h_2)$

$= (g_2 g_1, h_2 h_1)$ ($\because$ G & H are abelian)

$= (g_2,h_2)(g_1,h_1)$

$\Rightarrow G \oplus H$ are abelian

Remark: In general case, $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ is Abelian $\Leftrightarrow$ every G_i is abelian.

(Q) Find all subgroups of order 3 in $Z_9 \oplus Z_3$

(Sol^n) Subgrp of order 3 are cyclic $\Rightarrow$ elt of order 3

Let $(m,n) \in Z_9 \oplus Z_3$ of order 3

$O(m,n) = \text{lcm}(|m|, |n|)$

possibilities are

$(1,3) \Rightarrow m = 0, \; n = 1, 2$

$(3,1) \Rightarrow m = 3, 6, \; n = 0$

$(3,3) \Rightarrow m = 3, \; n = 1, 2$

($O(0) = 1 \rightarrow$ in Z_9

$\left.\begin{array}{l} O(1) = 3 \\ O(2) = 3 \end{array}\right]$ in Z_3

i.e $O(3) = 3 \rightarrow$ in Z_9

$O(0) = 1$ in Z_3

(Q) find all subgroups of order 3 in $Z_9 \oplus Z_3$

(Sol^n) Subgrp of order 3 are cyclic

$\Rightarrow$ elt of order 3

Let $(m,n) \in Z_9 \oplus Z_3$ of order 3

$$\left.\begin{matrix}(1,3)\\(3,1)\\(3,3)\end{matrix}\right\} \quad o(m,n) = lcm(|m|,|n|)$$

(case (i)) (1,3)

Z_9 has 1 elt of order 1 i.e 0

Z_3 has 2 elts of order 3 i.e 1, 2

$\Rightarrow m = 0,\ n = 1, 2$

case (ii) (3,1)

Z_9 has 2 elts of order 3 i.e 3, 6

$\Rightarrow m = 3, 6$

Z_3 has 1 elt of order 1 i.e 0

$\Rightarrow n = 0$

case (iii) (3,3)

Z_9 has 2 elts of order 3 i.e 3, 6

Z_3 " 2 elts of order 3 i.e 1, 2

$\therefore$ we have pairs

$(0,1)\ (0,2)$

$(3,0)\ (6,0)$

$(3,1)\ (3,2),\ (6,1)\ (6,2)$

$\therefore$ Distinct subgroups of order 3 are

$\langle (0,1) \rangle = \{(0,0), (0,1), (0,2)\}$

$\langle (3,0) \rangle = \{(0,0), (3,0), (6,0)\}$

$\langle (3,1) \rangle = \{(0,0), (3,1), (6,2)\}$

$\langle (3,2) \rangle = \{(0,0), (3,2), (6,1)\}$

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(10) Express $U(165)$ as an external direct product of ~~ ~~ cyclic additive groups of form Z_n.

(Sol^n) $U(165) = U(3\cdot5\cdot11) \approx U(3) \oplus U(5) \oplus U(11)$

$\approx Z_2 \oplus Z_4 \oplus Z_{10}$

$\left[\because U(st) \approx U(s) \oplus U(t) \quad (s,t)=1\right]$

(Q) In $Aut(Z_{20})$, determine how many elts of $Aut(Z_{20})$ have order 4. How many elts of order 2?

(Sol^n) $Aut(Z_{20}) \approx U(20) \approx$ ~~ ~~ $U(4) \oplus U(5)$

$\approx Z_2 \oplus Z_4$

To obtain number of elements of order 4.

$4 = lcm(1,4)$ in $Z_2 \oplus Z_4$

$lcm(2,4)$ in $Z_2 \oplus Z_4$

$(2,2)$ X

$(4,1)$ X

$(4,2)$ X

possible ~~ ~~ only $(1,4)$ $\Rightarrow$ $\phi(1) = 1$ in Z_2

$\phi(4) = 2$ in Z_2

$\Rightarrow$ 1&2 are elts

$\Rightarrow$ 2 elts in Z_2

Similarly $(2,4)$
$\Rightarrow$ 2 elts in $Z_2 \oplus Z_4$

$\therefore$ total number of elts order 4 are 4 elements $(2+2)$

Similarly it has 3 elements of order 2.

(Q) what is the largest order of any elt in $U(900)$

(Soln) $U(900) = U(2^2 \cdot 3^2 \cdot 5^2)$
$\approx Z_2 \oplus Z_6 \oplus Z_{20}$

The largest possible order is $lcm(2,6,20) = 60$

(Q) In $U(27)$, how many subgrp $U(27)$ has
$U(27) = U(3^3) \approx Z_{18}$

Now $18 = 3^2 \cdot 2$
By fundamental theorem of cyclic grps
the grup Z_{18} is cyclic & it has exactly one subgrp of order K where K is divisor of 18
Number of divisors of 18 are 6
$\tau(18) = \tau(3^2 \cdot 2) = (2+1)(1+1)$
$= 3 \cdot 2 = 6$

$\therefore$ there are 6 subgrps of Z_{18}, for each positive divisor of 18, and it is unique