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An extension of the Cauchy integral formula ①

let function f is analytic everywhere ^(Now formula for whole) inside and on a simple closed contour C , taken in positive sense. then we write the Cauchy integral formula as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} ds \quad - (1)$$

where z is in interior of C

and s denotes points on C .

Now, differentiating 'f' w.r.t z under the integral sign, we get

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds \quad - (2)$$

Now, how it comes

to verify $f'(z)$ exists & expression (2) is valid,

let d denote the smallest distance from z to points s on C

and use (1) to write

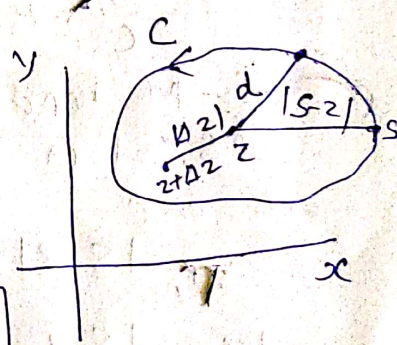
$$\begin{aligned} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{1}{\Delta z} \left[\frac{1}{2\pi i} \int_C \frac{f(s)}{s-(z+\Delta z)} ds - \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} ds \right] \\ &= \frac{1}{2\pi i} \int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right) \frac{f(s)}{\Delta z} ds \end{aligned}$$

where $0 < |\Delta z| < d$
(as limit $|\Delta z| \rightarrow 0$ here $d = d$)

then

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2}$$

$$= \frac{1}{2\pi i} \int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right) \frac{f(s)}{\Delta z} ds = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2}$$



$$\begin{aligned}
 \text{i.e. } & \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} \\
 &= \frac{1}{2\pi i} \int_C \left(\frac{1}{(s-z-\Delta z)} - \frac{1}{s-z} \right) \frac{f(s) ds}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} \\
 &= \frac{1}{2\pi i} \int_C \left[\frac{(s-z) - (s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2} \right] \frac{f(s) ds}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} \\
 &= \frac{1}{2\pi i} \int_C \frac{\Delta z f(s)}{(s-z-\Delta z)(s-z)^2} ds - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} \\
 &= \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z-\Delta z)(s-z)} - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} \\
 &= \frac{1}{2\pi i} \int_C \frac{(s-z) - (s-z-\Delta z)}{(s-z-\Delta z)(s-z)^2} f(s) ds \\
 &= \frac{1}{2\pi i} \int_C \frac{\Delta z f(s) ds}{(s-z-\Delta z)(s-z)^2}
 \end{aligned}$$

let M denote the maximum value of $|f(s)|$ on C .
 Δz , d is smallest distance from z to points S on C .

So, $|s-z| \geq d$ and $|\Delta z| < d$

$$\begin{aligned}
 \text{then } |s-z-\Delta z| &= |(s-z) - \Delta z| \\
 &\geq ||s-z| - |\Delta z|| \\
 &\geq d - |\Delta z| \\
 &> 0
 \end{aligned}$$

$$\begin{aligned}
 &\text{by } z_1, z_2 \in C \\
 &|z_1 - z_2| \leq |z_1 - z| + |z - z_2| \\
 &\leq |z_1| + |z_2| \\
 &\text{(Since } 0 < |\Delta z| < d \\
 &\Rightarrow d - |\Delta z| > 0)
 \end{aligned}$$

$$\left| \int_C \frac{\Delta z f(s) ds}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{|\Delta z| M}{(d-|\Delta z|)d^2} L$$

AS $|s-z|^2 \geq d$

where L is length of C . $\left(\int_C ds = L \right)$

$$\begin{aligned}
 \text{and } |z_1 - z_2| &\leq |z_1 - z| + |z - z_2| \\
 &\leq |z_1| + |z_2|
 \end{aligned}$$

Letting Δz tends to zero
then RHS tends to zero ~~(proof)~~

(2)

$$\therefore \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2} = 0$$

$$\Rightarrow \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^2}$$

$\therefore$ we get desired expression
for $f'(z)$ as eqⁿ (3)

Similarly, we get

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s) ds}{(s-z)^3} \quad (4)$$

By mathematical induction, we obtain formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s) ds}{(s-z)^{n+1}} \quad (n=1, 2, \dots) \quad (5)$$

$$\text{Also, } f^{(0)}(z) = \frac{0!}{2\pi i} \int_C \frac{f(s) ds}{(s-z)}$$

$$f^{(0)}(z) = f(z)$$

eqⁿ (5) is also valid when $n=0$, we get
Cauchy integral formula.

eqⁿ (5) can be written as

$$\int_C \frac{f(z) dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \quad (n=1, 2, \dots)$$

Example (1) If C is positively oriented unit circle $|z|=1$
and $f(z) = \exp(2z)$

then find $\int_C \frac{\exp(2z)}{z^4} dz$

$$\text{i.e. } \int_C \frac{f(z) dz}{z^4}$$

~~where~~ $z=0$

$|z| < 1$
(inside)

as letting Δz tends to zero
then RMS tends to zero ~~(proof)~~

(2)

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$$\text{then find } \int_C \frac{\exp(2z)}{z^4} dz$$

$$\text{i.e. } \int_C \frac{f(z) dz}{z^4}$$

~~exp~~ $\frac{1}{z^4}$

$|z| < 1$
(inside)

Here $z_0 = 0$ (interior)

$$\therefore \int \frac{\exp(2z)}{z^4} dz = \int \frac{f(z) dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0)$$

$$f(z) = \exp(2z)$$

$$f'(z) = e^{2z}(2)$$

$$f''(z) = e^{2z}(4)$$

$$f'''(z) = e^{2z}(8)$$

$$f'''(0) = 8$$

$$= \frac{8 \times 2\pi i}{3 \times 2}$$

$$= \frac{8\pi i}{3}$$

Exg ② let z_0 be any point interior to a positively oriented simple closed contour C . when $f(z) = 1$

$$\text{find } \int_C \frac{dz}{z-z_0}$$

$$\text{(Sol)} \text{ By formula } \int_C \frac{f(z) dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

$$\text{Here } n=0, f(z)=1$$

$$\int \frac{dz}{z-z_0} = \frac{2\pi i}{1!} f^0(z_0) = 2\pi i(1) = 2\pi i$$

$$f^0(z_0) = f(z_0)$$

$$\text{and } \int_C \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0) = 0 \quad (0: f'(z_0) \neq 1)$$

($n=1, 2, \dots$)