8.3 Convergence in Probability

Definition: Suppose $X_1, X_2, \ldots$ is a sequence of random variables. Then X_n converges in probability to a number a if, for every $\varepsilon > 0$

$$\lim_{n \to \infty} P[|X_n - a| \geq \varepsilon] = 0$$

In other words, if for every $\delta > 0$ and $\varepsilon > 0$, there exists n_0 such that

$$P[|X_n - a| \geq \varepsilon] \leq \delta \quad \forall \quad n \geq n_0$$

Notation $\quad X_n \xrightarrow{P} a \quad$ as $n \to \infty$

8.4 Weak Law of Large Numbers

Statement: Suppose $X_1, X_2, \ldots X_n$ are i.i.d. random variables with mean μ and variance σ^2. Then the sample mean $B_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$ converges to the actual mean μ in probability:

$$P[|B_n - \mu| \geq \varepsilon] \to 0 \quad \text{as } n \to \infty$$

for every $\varepsilon > 0$

Proof : $B_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$

$$E(B_n) = E\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n}\left[E(X_1) + E(X_2) + \cdots + E(X_n)\right]$$

$$= \frac{1}{n}\left[\mu + \mu \cdots + \mu\right] \qquad [\because E(X_i) = \mu]$$

$$= \mu$$

$$Var(B_n) = Var\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n^2} \cdot Var\left[X_1 + X_2 + \cdots + X_n\right]$$

$$= \frac{1}{n^2} \left[Var(X_1) + Var(X_2) \cdots + Var(X_n)\right] \qquad [\because X_i\text{'s are iid})$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

By Chebyshev's inequality, for any $\varepsilon > 0$

$$P\left[\,|B_n - E(B_n)| \geq \varepsilon\right] \leq \frac{Var(B_n)}{\varepsilon^2}$$

$$\Rightarrow P\left[\,|B_n - \mu| \geq \varepsilon\right] \leq \frac{\sigma^2}{n^2\varepsilon^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow P\left[\,|B_n - \mu| \geq \varepsilon\right] \to 0 \text{ as } n \to \infty \text{ for any } \varepsilon > 0$$

Hence proved

8.4.1 Another form of Weak law of large numbers

Let $\{X_n\}$ be a sequence of random variables and $S_n = \sum_{i=1}^{n} X_i$ with $B_n = Var(S_n) < \infty$

Then for $\varepsilon > 0$

$$\lim_{n\to\infty} P\left[\left| \frac{S_n}{n} - E\left(\frac{S_n}{n}\right) \right| < \varepsilon \right] = 1 \quad -(1)$$

provided $\frac{B_n}{n^2} \to 0$ as $n \to \infty$

Remark 1. We can write (1) as

$$\lim_{n\to\infty} P\left[\left| \bar{X}_n - \mu \right| < \varepsilon \right] = 1 \text{ if } \lim_{n\to\infty} \frac{B_n}{n^2} = 0$$

where $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$, $E(\bar{X}_n) = \mu$

Remarks 2. Sufficient condition for WLLN to hold is $\lim_{n\to\infty} \frac{B_n}{n^2} \to 0$. But it is not the necessary condition

Remark 3. $\lim_{n\to\infty} \frac{B_n}{n^2} = 0$ becomes the necessary and sufficient condition for the WLLN to hold if $\{X_n\}$ is a sequence of uniformly bounded variates.

Remark 4. Weak law of large numbers is also stated as law of large numbers

8.5 Strong Law of Large Numbers

Statement: Let $\{X_n\}$ be a sequence of i.i.d random variables and $E(X_i) = \mu$

Also let $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$

Then $\bar{X}_n$ converges almost surely to μ

i.e $P[\lim_{n\to\infty} \bar{X}_n = \mu] = 1$

8.6 Central Limit Theorem (CLT)

This theorem provides a simple method for computing approximate probabilities for the sums of independent random variables. Besides this it states that under certain conditions the sum of a large number of random variables is approximately normal.

8.6.1 Statement (CLT): Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables each with finite mean μ and variance σ^2.

Then the distribution of

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma}$$

tends to standard normal as $n \to \infty$. That is

$$P[Z_n \le x] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du \quad \text{as } n \to \infty$$

Proof: Consider

$$M_{Z_n}(t) = M_{\frac{\bar{x}_n - \mu}{\sigma/\sqrt{n}}}(t)$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} M_{\bar{x}_n}\left(\frac{\sqrt{n}t}{\sigma}\right)$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} \cdot M_{n\bar{x}_n}\left(\frac{t}{\sigma\sqrt{n}}\right) \quad -(1)$$

Since $n\bar{x}_n = x_1 + x_2 + \cdots + x_n$, then

$$M_{n\bar{x}_n}\left(\frac{t}{\sigma\sqrt{n}}\right) = \left[M_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

[$\because$ $x_1, x_2 \cdots x_n$ are i.i.d]

~~Take logarithm on both si~~

$\therefore$ (1) becomes

$$M_{Z_n}(t) = e^{-\frac{\sqrt{n}\mu t}{\sigma}} \left[M_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

Take logarithm on both sides, we get

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\mu t}{\sigma} + n \ln M_x\left(\frac{t}{\sigma\sqrt{n}}\right)$$

On expanding $M_x\left(\frac{t}{\sigma\sqrt{n}}\right)$ as a power series in t, we obtain

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\mu t}{\sigma} + n \ln\left[1 + \mu_1' \frac{t}{\sigma\sqrt{n}} + \mu_2' \frac{t^2}{2\sigma^2 n} + \mu_3' \frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]$$

where μ_1', μ_2', μ_3' are the moments about origin of the population distribution

For sufficiently large n, we can use the expansion of $\ln(1+x)$ as a power series in x, getting

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\,\mu t}{\sigma} + n\left\{ \left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right] - \frac{1}{2}\left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]^2 + \frac{1}{3}\left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]^3 + \cdots \right\}$$

Now collecting the powers of t, we get

$$\ln M_{Z_n}(t) = \left(-\frac{\sqrt{n}\,\mu}{\sigma} + \frac{\sqrt{n}\,\mu_1'}{\sigma}\right)t + \left(\frac{\mu_2'}{2\sigma^2} - \frac{\mu_1'^2}{2\sigma^2}\right)t^2 + \left(\frac{\mu_3'}{6\sigma^3\sqrt{n}} - \frac{\mu_1'\mu_2'}{2\sigma^3\sqrt{n}} + \frac{\mu_1'^3}{3\sigma^3\sqrt{n}}\right)t^3 + \cdots$$

Since $\mu_1' = \mu$ and $\mu_2' - \mu_1' = \sigma^2$, so above expression reduces to

$$\ln M_{Z_n}(t) = \frac{1}{2}t^2 + \left(\frac{\mu_3'}{6} - \frac{\mu_1'\mu_2'}{2} + \frac{\mu_1'^3}{6}\right)\frac{t^3}{\sigma^3\sqrt{n}} + \cdots$$

As the coefficient of t^3 is a constant times $\frac{1}{\sqrt{n}}$ then in general coeff. of t^r is constant times $\frac{1}{\sqrt{n^{r-2}}}$, for $r \geq 2$

$$\therefore \lim_{n\to\infty} \ln M_{Z_n}(t) = \frac{1}{2}t^2$$

$$\Rightarrow \lim_{n \to \infty} M_{Z_n}(t) = e^{t^2/2}$$

which is the m.g.f of standard normal distribution

~~Hence proved the theorem~~

By the uniqueness property of m.g.f

The distribution of $Z_n = \dfrac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ tends to standard normal as $n \to \infty$

Hence proved the theorem

$\boxed{\text{EXAMPLES}}$

Example 1. A soft drink vending machine is set so that the amount of drink dispensed is a random variable with a mean of 200 millilitres and a standard deviation of 15 millilitres. What is the probability that the average (mean) amount dispensed in a random sample of size 36 is atleast 204 millilitres?

Sol: Let X = amount of drink dispensed from a soft drink vending machine

Given $\mu = E(X) = 200$ and $\sigma^2 = Var(X) = 15$

$\therefore$ A random sample of size 36 is drawn

So $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_{36}}{36}$

$\therefore \quad \mu_{\bar{X}} = 200$ and $Var(\bar{X}) = \dfrac{15}{36}$

Hence standard deviation

$\sigma_{\bar{X}} = \dfrac{15}{\sqrt{36}}$

According to CLT, this distribution $(\bar{X})$ is approximately normal

$\therefore P[\bar{X} \geq 204] = P\left[\dfrac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \geq \dfrac{204-200}{15/\sqrt{36}}\right]$

$= P[Z \geq 1.6]$

$= 0.5000 - 0.4452$

$= 0.0548$

Example 2. The lifetime of a special type of battery is a random variable with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assuming a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained

Sol: Let X_i denote the lifetime of the ith battery to be put in use

Also $\mu = E(X_i) = 40$ and $\sigma^2 = Var(X_i) = 400$

$\sigma = 20$

Then we want to find $P[X_1 + X_2 + \dots + X_{25} \geq 1100]$ which is approximated as follows using CLT

$P[\cancel{X_1 + X_2 + \dots + X_{25} = 1000}$

$$P\left[\frac{X_1 + X_2 + \dots + X_{25} - 25\mu}{\sigma\sqrt{25}} \geq \frac{1100 - 1000}{20 \times 5}\right]$$

$$\simeq P\left[Z \geq \frac{100}{5 \times 20}\right] = P[Z \geq 1]$$

$$= 1 - \Phi(1)$$

$$= 0.1587$$

Example 3. Let X be the number of times that a fair coin flipped 40 times, lands heads. Find the probability that $X = 20$. Use normal approximation to compare it with exact solution.

Sol: X is the number of times that a fair coin lands heads

Given $n = 40$, $\theta = \frac{1}{2}$

Since X follows binomial distribution

$\therefore \mu = n\theta = 20$ & $\sigma^2 = n\theta(1-\theta) = 10$

Now $P[X=20] = P[19.5 < X < 20.5]$

$$= P\left[\frac{19.5-20}{\sqrt{10}} < \frac{X-\mu}{\sigma} < \frac{20.5-20}{\sqrt{10}}\right]$$

$$= P[-0.16 < Z < 0.16] \quad [\text{Using CLT}]$$

$$\simeq \phi(0.16) - \phi(-0.16) \quad -(1)$$

By symmetry of standard normal distribution

$$\phi(-0.16) = 1 - \phi(0.16)$$

$\therefore (1) \Rightarrow \quad P[X=20] \simeq 2\phi(0.16) - 1$

$$\simeq 2\cdot(0.5637) - 1$$

$$\simeq 0.1272$$

The exact value of $P[X=20] = {}^{40}C_{20}\left(\frac{1}{2}\right)^{40}$

$$= 0.1268$$

Example 4. Let X_i ; $i=1,2,--10$ be independent random variables each being uniformly distributed over $(0,1)$. Estimate $P\left[\sum_{i=1}^{10} X_i > 7\right]$ using CLT

Solution: Given ~~each~~ X_i ; $i=1,2,--10$ are independ random variable and follows uniform distribution over $(0,1)$

$\therefore E(X_i) = \frac{1}{2}$ and $Var(X_i) = \frac{1}{12}$

~~Therefore by CLT~~

~~$P\left[\sum_{i=1}^{10} X_i > 7\right] = P\left[\sum_{i=1}^{10} X_i -\right.$~~

Now $\mu = E(X_1 + X_2 + - - + X_{10}) = 5$

$\sigma^2 = Var[X_1 + X_2 + - - + X_{10}] = \frac{10}{12}$

Using CLT, we have

$$P\left[\sum_{i=1}^{10} X_i > 7\right] = P\left[\frac{\sum_{i=1}^{10} X_i - \mu}{\sigma} > \frac{7-5}{\sqrt{\frac{10}{12}}}\right]$$

$$\cong 1 - \phi(2.2)$$

$$= 0.0139$$