Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ & hence find $\Gamma(-\frac{1}{2})$

Proof:-

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

put $x^n = y \rightarrow x = y^{\frac{1}{n}}$

$dx = \frac{1}{2} y^{\frac{1}{n}-1} dy$

then $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}} dy$

put $n = \frac{1}{2}$

$$\frac{1}{2}\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-y^2} dy \quad \text{------ (I)}$$

put $y = \alpha t$

$dy = \alpha dt$

~~in (I) multiply by $e^{-\alpha^2}$~~

$$\frac{1}{2}\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-\alpha^2 t^2} \alpha dt \quad \text{------ (II)}$$

we multiplying in (II) by $e^{-\alpha^2}$ & integerate w.r.t α b/w the limits 0 to ∞

$$\frac{1}{2}\Gamma(\tfrac{1}{2})\int_0^\infty e^{-\alpha^2} d\alpha = \int_0^\infty \left[\int_0^\infty e^{-\alpha^2(1+t^2)} \alpha d\alpha\right] dt$$

$$= \int_0^\infty \left[-\frac{e^{-\alpha^2(1+t^2)}}{2(1+t^2)}\right]_0^\infty dt$$

$$= \frac{1}{2}\int_0^\infty \frac{1}{(1+t^2)} dt = \frac{1}{2}\tan^{-1}(t)\Big]_0^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{1}{2}\frac{\pi}{2}$$

using (I)

$$\frac{1}{2}\Gamma\tfrac{1}{2} \cdot \frac{1}{2}\Gamma\tfrac{1}{2} = \frac{1}{4}\pi$$

$$[\Gamma(\tfrac{1}{2})]^2 = \pi$$

$$\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}} \quad \text{Ans}$$

we know that

$$n\Gamma n = \Gamma n+1$$

put $n = -\frac{1}{2}$

$$-\frac{1}{2}\Gamma{-\tfrac{1}{2}} = \Gamma\tfrac{1}{2}$$

$$\Gamma(-\tfrac{1}{2}) = -2\Gamma\tfrac{1}{2} = -2\sqrt{\pi}$$

$$\therefore \boxed{\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}} \quad \text{Ans}$$

Qu: Show that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \frac{\pi}{\sqrt{2}}$

(ii) $= \frac{\pi}{2}\frac{1}{\sin\frac{\pi}{4}}$

Proof:- $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\frac{\sin^{\frac{1}{2}}\theta}{\cos^{\frac{1}{2}}\theta}\,d\theta = \int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta$

[using formula $2\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$]

$$\therefore = \frac{\Gamma(\frac{\frac{1}{2}+1}{2})\Gamma\frac{-\frac{1}{2}+1}{2}}{2\Gamma\frac{(\frac{1}{2}-\frac{1}{2}+2)}{2}} = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{2\Gamma(1)} = \frac{1}{2}\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) \quad \text{--- (1)}$$

Now using duplication formula

$$\Gamma m\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

put $m=\frac{1}{4}$

$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) = \frac{\sqrt{\pi}}{2^{2\times\frac{1}{4}-1}}\Gamma 2\times\tfrac{1}{4}$$

$$= \frac{\sqrt{\pi}}{2^{\frac{1}{2}-1}}\Gamma(\tfrac{1}{2}) \qquad [\because \Gamma\tfrac{1}{2} = \sqrt{\pi}]$$

$$\frac{1}{2}\Gamma\tfrac{1}{4}\Gamma\tfrac{3}{4} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{\sqrt{2}\times 2} = \frac{\pi}{\sqrt{2}}$$ put in (1)

$$\boxed{\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \frac{1}{2}\Gamma(\tfrac{1}{4})\Gamma\tfrac{3}{4} = \frac{\pi}{\sqrt{2}}}$$

(ii) Similarly second part:

using $\Gamma m\Gamma 1-m = \frac{\pi}{\sin m\pi}$ $\quad m=\frac{3}{4}$ so

$\Gamma\frac{3}{4}\Gamma(\frac{1}{4}) = \frac{\pi}{\sin\frac{3\pi}{4}}$

(1) become $= \frac{1}{2}\frac{\pi}{\sin\frac{3\pi}{4}} = \frac{\pi}{2}\frac{1}{\sin\frac{3\pi}{4}}$

Que.: show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

Proof :- LHS

$$I = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta \cos^0\theta\, d\theta$$

using $2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

$$I = \frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} \times \frac{1}{2}\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{1}{2}\frac{\cancel{\Gamma\left(\frac{3}{4}\right)}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \times \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{3}{4}\right)}}$$

$$= \frac{\cancel{1}}{\cancel{4}}\frac{\cancel{\Gamma\left(\frac{1}{4}\right)}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\cancel{\frac{1}{4}}\cancel{\Gamma\left(\frac{1}{4}\right)}}$$

$$= \pi$$

$$\therefore \boxed{\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi}$$

Legendre's Duplication formula : $\boxed{\Gamma(m)\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$

we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\,d\theta$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\,d\theta \cdot \sin\theta\cos\theta$$

$$\therefore \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \quad \text{--- (a)} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$$

put $n = \frac{1}{2}$ in (a)

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta \quad \text{------- (1)}$$

put $n = m$ in (a)

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta$$

put $2\theta = \phi$
$d\theta = \frac{d\phi}{2}$

$$\therefore \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi\,d\phi$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi$$

$$2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi, \quad \phi \leftrightarrow \theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta \quad \text{——— (2)}$$

from (1) & (2) we get $\frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma(m+\frac{1}{2})} = 2^{2m-1}\frac{\Gamma m\,\Gamma m}{\Gamma(2m)}$ $\quad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right.$

$$\boxed{\Gamma(m)\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

Power Series

A series of the form $\boxed{\sum_{n=0}^{\infty} a_n x^n}$ - is said to a power series or --- (1)
(real power series) where $a_0, a_1, a_2 \cdots a_n$ are called real coefficients free from 'x' and 'x' is the real variable.

if we shift the origin to x_0 with the change of variable $x - x_0$ then eqn (1) can be written as

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Note :— For all values of the coefficients every power series converges for $x = 0$.

Analytic function : A function $f(x)$ is said to analytic at x_0 if it is differentiable at x_0 as well as every point of some neighbourhood of x_0.

OR

A function $f(x)$ which can be extended in Taylor's Series on interval containing the point $x = x_0$. The series converges to $f(x)$ $\forall$ x in the interval of convergence.

$f(x)$. can be written as in Taylor's Series such as $\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x - x_0)$

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Examples:

- All polynomial functions
- e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$ are analytic everywhere
- A rational no: is analytic except those values of x at which denominator is zero

e.g $\frac{x}{x^2-3x+2}$ $\left(\text{or } \frac{x}{(x-1)(x-2)}\right)$ is analytic everywhere except at $x=1$ & $x=2$.

Taylor's Series: A function $f(x)$ is analytic at all points inside c with its centre at the point x_0 & radius R then at each point inside 'c'.

$$\boxed{f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^n(x_0)}{n!}(x-x_0)^n}$$

Taylor's Series

if $x=0$

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots \frac{x^n}{n!}f^n(0)}$$

→ Maclaurin Series

On Complex Variable: A function $f(z) = u+iv$ is said to analytic function of z if its real (u) & img (v) parts satisfying the Cauchy-Reimann Equations / Conditions

CR eqns
$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

- Analytic function is also known as

Holomorphic or Regular or monogenic

- f(z) is analytic in a domain if it is analytic at every point of domain.

Ordinary & Singular Points

Ordinary Point: A point $x=x_0$ is called an ordinary point of the eqn. $y''+P(x)y'+Q(x)y=0$ --- ① if both the functions $P(x)$ & $Q(x)$ are analytic at $x=x_0$

OR

we can say $x=x_0$ is an ordinary point of ① if the denominator of $P(x)$ & $Q(x)$ does not vanish at $x=x_0$ ie ($P\neq\infty$; $Q\neq\infty$)

- if $x=0$ is the ordinary point then it is called $\boxed{\text{Regular Point.}}$

Singular Point: if $x=x_0$ is not an ordinary point of eqn ① then it is called a singular point. There are two types of singular points. (i) regular singular point
(ii) irregular singular point.

To check singular point: $\boxed{P=\infty \text{ or } Q=\infty \text{ or both}}$

(i) **Regular singular point:** if both $(x-x_0)P(x)$ & $(x-x_0)^2Q(x)$ are analytic at $x=x_0$ $(\neq\infty)$. Then $x=x_0$ is a regular singular point of eqn①.

(ii) **Irregular Singular point:** A singular point which is not regular is called an irregular singular point. for this $(x-x_0)P(x)$ and $(x-x_0)^2Q(x) = \infty$ at $x=x_0$.

Note:- Generally, If at $x=x_0$ the function is not differentiable then the point is called singular point of that function.

e.g $\frac{1}{x-2}$ has a singular point at $x=2$.

P(4)

EX1: Det. whether $x=0$ is an ordinary point or regular singular point of the diff. eqn. $2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$

Soln: Dividing the eqn by $2x^2$, we get

$$\frac{d^2y}{dx^2}+\frac{7}{2}\left(\frac{x+1}{x}\right)\frac{dy}{dx}-\frac{3}{2x^2}y=0 \quad ---(1)$$

Comparing (1) with standard eqn $y''+P(x)y'+Qy=0$ --(2)

we have $P(x)=\frac{7}{2}\left(\frac{x+1}{x}\right)$ and $Q(x)=-\frac{3}{2x^2}$

At $x=0$:

$P(x)=\infty$ and $Q(x)=\infty$ ie not defined.

So, P & Q are not analytic.

$\therefore$ $x=0$ is not an ordinary point.

So $x=0$ is a singular point.

Now $(x-0)P(x)=\frac{7}{2}(x+1)=\frac{7}{2}\neq\infty$ at $x=0$

& $(x-0)^2Q(x)=-\frac{3}{2}\neq\infty$ at $x=0$

So at $x=0$ $(x-0)P(x)$ & $(x-0)^2Q(x)$ are analytic

$\therefore$ $x=0$ is a regular singular point.