Order of a Permutation

Q. Find the order of following permutations

(i) $(1\,2\,4)(3\,5\,7) \to$ order $= LCM\{3,3\} = 3$

(ii) $(1\,2\,4)\,(3\,5) \to$ order $= LCM\{3,2\} = 6$

(iii) $(1\,2\,4)(3\,5\,7\,8) \to LCM(3,4) = 12$

(iv) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$

$(1\,2)(3\,5\,6)(4)$

order $= LCM\{2,3,1\} = 6$

(v) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

$(1\,7\,5\,3)(2\,6\,4)$

order $= LCM\{4,3\}$
$= 12$

Product of 2-Cycles

Th^m Every permutation in S_n, $n > 1$, is a product of 2-cycles.

Proof:

First, we can see that identity permutation can be expressed as $(12)(12)$.

Now, we know that every permutation α can be written as a cycle or product of disjoint cycles.

$(12) \to$ length 2
$(12)^2 = I = (12)(12)$
2-cycle = cycle of length 2

$$\alpha = (a_1 a_2 \cdots a_m)(b_1 b_2 \cdots b_n) \cdots (c_1 c_2 \cdots c_k)$$

this is same as:

$$\alpha = (a_1 a_m)(a_1 a_{m-1}) \cdots (a_1 a_2)(b_1 b_n)(b_1 b_{n-1}) \cdots (b_1 b_2)$$
$$\cdots (c_1 c_k)(c_1 c_{k-1}) \cdots (c_1 c_2)$$

$(1\,2\,3) = (1\,3)(1\,2)$
check!

$(1234) = (14)(13)(12)$

$(12345) = (15)(14)(13)(12)$

$\therefore$ every permutation can be written as product of 2-cycles.

$\varepsilon = I =$ Identity Permutation

Lemma:

If ε is an identity permutation and $\varepsilon = \beta_1 \beta_2 \cdots \beta_r$, where the β's are two cycles, then r is even.

Examples

$\varepsilon = (12)(12) \to r = 2$.

$\varepsilon = (12)(12)(34)(34) \to r = 4$.

Theorem: No. of 2-cycles are always even or always odd.

If a permutation α can be expressed as a product of an even number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even number of 2-cycles.

that is, if $\alpha = \beta_1 \beta_2 \cdots \beta_r$ and $\alpha = \gamma_1 \gamma_2 \cdots \gamma_s$, where β's and γ's are 2-cycles. Then r and s are both even or both odd.

$(1\,2\,3) = (1\,3)\,(1\,2) \rightarrow 2\text{(even)}$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

$(1\,2\,3) = (1\,2)(2\,3) \rightarrow 2\text{(even)}$

$(1\,2\,3\,4\,5) = (1\,5)(1\,4)\,(1\,3)(1\,2) \rightarrow 4\text{ (even)}$

$(1\,2\,3\,4\,5) = (5\,4)\,(5\,2)(2\,1)(2\,5)(2\,3)(1\,3)$

$\rightarrow 6\text{ (even)}$

Proof: Given that

$$\alpha = \beta_1 \beta_2 \cdots \beta_r \quad \& \quad \alpha = \gamma_1 \gamma_2 \cdots \gamma_s$$

$$\therefore \beta_1 \beta_2 \cdots \beta_r = \gamma_1 \gamma_2 \cdots \gamma_s$$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s = \beta_1 \beta_2 \cdots \beta_r$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s \beta_r^{-1} = \beta_1 \beta_2 \cdots \beta_{r-1} \beta_r \beta_r^{-1}$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s \beta_r^{-1} = \beta_1 \beta_2 \cdots \beta_{r-1}$ $\boxed{a a^{-1} = e}$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s \beta_r^{-1} \beta_{r-1}^{-1} = \beta_1 \beta_2 \cdots \beta_{r-2}$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s \beta_r^{-1} \beta_{r-1}^{-1} \cdots \beta_1^{-1} = \varepsilon$

$\because$ 2-cycle is its own inverse.

$(12) \to$ order 2.
$(12)(12) = \varepsilon$
$(12)^{-1} = (12)$

$|a| = 2 \Rightarrow a^2 = e$.
$(12)^2 = \varepsilon \Rightarrow \underline{(12)}\underset{b}{\underline{(12)}} = \varepsilon$.
$a \cdot b = e$. $(12)^{-1} = (12)$

$a^2 = e \Rightarrow a = a^{-1}$

$\gamma_1 \gamma_2 \cdots \gamma_s \beta_r \beta_{r-1} \cdots \beta_1 = \varepsilon$

$\Rightarrow \gamma_1 \gamma_2 \cdots \gamma_s \beta_r \beta_{r-1} \cdots \beta_2 \beta_1 = \varepsilon$

Lemma, If $\varepsilon = \beta_1 \beta_2 \cdots \beta_r$, then r is even.

$\Rightarrow (r+s)$ must be even

$\Rightarrow r$ and s are both even or both odd.

Even and Odd Permutations:

A permutation that can be expressed as a product of an even number of 2-cycles is called an even permutation.

A permutation that can be expressed as a product of an odd number of 2-cycles is called an odd permutation.

Example: (i) $\varepsilon = (12)(12)$ is an even permutation

(ii) $(1234) = (14)(13)(12)$ is an odd permutation

Thm The set of all even permutations in S_n form a subgroup of S_n.

Proof:- ① Closure Property

Let α and β be two even permutations then α & β can be expressed as

$\alpha = a_1 a_2 \cdots a_r$. $\beta = b_1 b_2 \cdots b_s$

where a's and b's are two-cycles and r & s are even.

Now $\alpha\beta = a_1 a_2 \cdots a_r b_1 b_2 \cdots b_s$

$\therefore$ $\alpha\beta$ is a product of $(r+s)$ 2-cycles.

and $r+s$ is even as r & s are even.

$\Rightarrow$ $\alpha\beta$ is an even permutation.

(2) <u>Associativity</u>

Composition is always associative.

(3) <u>Identity</u>

Identity permutation ε is always an even permutation

(4) <u>Inverse:</u>

Let α be an even permutation

then $\alpha = a_1 a_2 \cdots a_\ell$, where a_i's all 2-cycles.

$\alpha^{-1} = (a_1 a_2 \cdots a_\ell)^{-1} = a_\ell^{-1} a_{\ell-1}^{-1} \cdots a_2^{-1} a_1^{-1}$

$= a_\ell a_{\ell-1} \cdots a_2 a_1$ ($\because$ every 2-cycle has its own inverse.)

$\Rightarrow$ α^{-1} is an even permutation as ℓ is even.

$\therefore$ set of even permutations is a subgroup of S_n.

<u>Alternating Group of Degree n</u>

The group of even permutations of n symbols $1, 2, 3, \ldots, n$ is denoted by A_n and is called the alternating group of degree n.

Theorem: For $n>1$, A_n has order $\frac{n!}{2}$

Proof: Let U and V be the set of all even permutations and set of all odd permutations of S_n

$\therefore |S_n| = |U| + |V|$ ——— (1)

Now $(12)U = \{(12)\sigma \mid \sigma \text{ is an even permutation}\}$

$kA = \{ka \mid a \in A\}$

$\Rightarrow (12)U \subseteq V$

$\Rightarrow |(12)U| \leq |V|$

$\Rightarrow |U| \leq |V|$ ——— (2)

$\sigma = a_1 a_2 \cdots a_\ell$

$(12)\sigma = (12)a_1 a_2 \cdots a_\ell$ ℓ even

($\ell+1$ odd)

Now $(12)V = \{(12)\phi \mid \phi \text{ is an odd permutation}\}$

$\Rightarrow (12)V \subseteq U$ $\Rightarrow |(12)V| \leq |U|$

$\Rightarrow |V| \leq |U|$ ——— (3)

from eqn (2) & (3), $|U| = |V|$

using it in eqn (1),

$|U| + |U| = |S_n|$

$\Rightarrow 2|U| = n!$

$\Rightarrow |U| = \frac{n!}{2}$

$\Rightarrow |A_n| = \frac{n!}{2}$ $\left[\because U = A_n = \text{set of all even permutations}\right.$

Chapter-3 Q.13

Given that H is closed.

Given that If $a^{-1} \notin H$, then $a \notin H$

$\therefore a^{-1} \notin H \Rightarrow a \notin H$

TS $a \in H \Rightarrow a^{-1} \in H$.

let $a \in H$.

TS $a^{-1} \in H$

suppose $a^{-1} \notin H \Rightarrow \underline{a \notin H}$ $\times$.

$\therefore a^{-1} \in H$.

Q. (6) $|g| = m$

To S: n divides m.

By division algorithm

$m = nq + r$, where $0 \le r \le n-1$

Chapter 2

Q. (27) TS $(a^{-1}ba)^n = a^{-1}b^n a \ \forall n \in \mathbb{Z}$

Case-I: When $n=0$

$(a^{-1}ba)^0 = e$

and $a^{-1}b^0 a = a^{-1}ea = a^{-1}a = e$

$\therefore (a^{-1}ba)^n = a^{-1}b^n a$ for $n=0$.

Case-II: $n > 0$

$$(a^{-1}ba)^n = (a^{-1}ba)(a^{-1}ba)\cdots(a^{-1}ba) \; \{n \text{ times}\}$$

$$= (a^{-1}b)(aa^{-1})\,b(aa^{-1})\cdots(aa^{-1})\,b\,a \; \{n \text{ times}\}$$

$$= a^{-1}\underbrace{b\cdot b\cdot b\cdots b}_{n\text{ times}}\,a \quad [\because aa^{-1}=e]$$

$$= a^{-1}b^n a$$

Case – III When $n<0$

When $n<0 \Rightarrow -n>0$

$$\therefore (a^{-1}ba)^{-n} = a^{-1}b^{-n}a \quad \text{[from Case II]}$$

$$\Rightarrow \{(a^{-1}ba)^{-n}\}^{-1} = \{a^{-1}b^{-n}a\}^{-1}$$

$$\Rightarrow (a^{-1}ba)^n = a^{-1}(b^{-n})^{-1}(a^{-1})^{-1}$$

$$\Rightarrow (a^{-1}ba)^n = a^{-1}b^n a$$

$$\therefore (a^{-1}ba)^n = a^{-1}b^n a \quad \forall n\in\mathbb{Z}.$$

$(ab)^{-1} = b^{-1}a^{-1}$

(Socks–Shoes Property) Draw an analogy between the statement ~~(ab)21 5 b21 a21~~ and the act of putting on and taking off your socks and shoes. Find distinct nonidentity elements a and b from a non-Abelian group such that (ab)21 5 a21 b21. Find an example that shows that in a group, it is possible to have ~~(ab)22 Z b22 a22.~~ What would be an appropriate name for the group property (abc)21 5 c21 b21 a21?

$(ab)^{-1} = a^{-1}b^{-1}$

and shoes. Find distinct nonidentity elements a and b from a non-Abelian group such that (ab)21 5 a21 b21. Find an example that shows that in a group, it is possible to have (ab)22 Z b22 a22. What would be an appropriate name for the group property (abc)21 5 c21 b21 a21?

$(ab) = a \ b$

$(ab)^{-2} \neq b^{-2} a^{-2}$

$(abc)^{-1} = c^{-1} b^{-1} a^{-1}$

Sol^n

$$(ab)^{-1} = b^{-1} a^{-1}$$

The inverse of putting on your socks and then putting on your shoes is taking off your shoes first and then taking off your socks.

Non-abelian. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$a = i, b = j, \quad ab = k.$

$b^2 = j^2 = -1, \ (b^2)^{-1} = -1$

$\Rightarrow b^{-2} = -1.$

$a^2 = i^2 = -1 \Rightarrow (a^2)^{-1} = -1 \Rightarrow a^{-2} = -1$

$(ab)^2 = k^2 = -1 \Rightarrow \{(ab)^2\}^{-1} = (-1)^{-1} = -1$

$\Rightarrow (ab)^{-2} = -1.$

But $b^{-2} a^{-2} = (-1)(-1) = 1$

$\therefore (ab)^{-2} \neq b^{-2} a^{-2}.$

$(abc)^{-1} = c^{-1} b^{-1} a^{-1} \rightarrow$ Alterbiate name

$(ab)^{-1} = b^{-1} a^{-1} \rightarrow$ Socks- Shoes Property.

$A = \{1, 2, 3, 4, 5\}$

$$(1\ 2) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix}$$

$$(2\ 1) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix}$$

$\therefore \quad (12) = (21)$

$(13) = (31), \quad (34) = (43)$

$O\{(1\ 2)\} = 2 \rightarrow (12)(12) = \varepsilon$

and $(12)^{-1} = (12)$