Theory of Real functions

B.Sc (H) Mathematics 18/8/2020

More Examples on limit of functions:-

(f) Show that $\lim_{x \to c} x^3 = c^3$.

Let $f(x) := x^3$, $x \in \mathbb{R}$

Consider

$$|f(x) - c^3| = |x^3 - c^3|$$
$$= |(x-c)(x^2+c^2+xc)|$$
$$= |x-c|\,|x^2+c^2+xc|$$
$$(\because |ab| = |a|\,|b|)$$
$$\leq |x-c|(|x|^2+|c|^2+|x||c|) \quad \text{--- (1)}$$

let $|x-c|<1$ then

$|x|-|c| \leq ||x|-|c|| \leq |x-c| < 1$

$\Rightarrow$ $|x|-|c| < 1$ $\Rightarrow$ $|x| < |c|+1$

$\therefore$ (1) $\Rightarrow$

$$|f(x) - c^3| \leq |x-c|.(|x|^2+|c|^2+|x||c|)$$
$$< |x-c|((|c|+1)^2+|c|^2+(|c|+1)|c|)$$
$$= |x-c|(3|c|^2+1+3|c|)$$

$\Rightarrow \quad |f(x) - c^3| < |x-c| (3c^2 + 3|c| + 1)$

Then $\quad |f(x) - c^3| < |x - c| (3c^2 + 3|c| + 1) < \epsilon \quad$ — (2)

provided

$$|x - c| < \frac{\epsilon}{3c^2 + 3|c| + 1}$$

Thus, choose $\quad \delta = \inf \left\{ 1, \frac{\epsilon}{3c^2 + 3|c| + 1} \right\}$

$\therefore \quad |x - c| < \delta = \inf \left\{ 1, \frac{\epsilon}{3c^2 + 3|c| + 1} \right\} \leq 1$

$$\leq \frac{\epsilon}{3c^2 + 3|c| + 1}$$

$\Rightarrow \quad |x - c| < 1 \quad$ and $\quad |x - c| < \frac{\epsilon}{3c^2 + 3|c| + 1}$

Hence (2) $\Rightarrow$

$$|f(x) - c^3| < \epsilon$$

$$\therefore \quad \lim_{x \to c} f(x) = c^3.$$

(8) Show that $\lim\limits_{x \to 4} x + 1 = 5$

let $\epsilon > 0$ be given,

let $f(x) := x+1$, $L = 5$

Consider $|f(x) - L| = |x+1-5|$

$= |x-4| < \epsilon$

Provided $|x-4| < \delta$ and $\delta = \epsilon$.

$\therefore$

$\therefore$ choose $\delta = \epsilon$ then

$0 < |x-4| < \delta \Rightarrow |f(x) - L| < \epsilon$.

(h) Show that $\lim_{x \to -2} x^2 = 4$. (H.W.)

(i) Show that $\lim_{x \to 4} \sqrt{x+1} = \sqrt{5}$, $x > 0$

let $\epsilon > 0$ be given, $f(x) := \sqrt{x+1}$
$L = \sqrt{5}$

Then $|f(x) - L|$

$= |\sqrt{x+1} - \sqrt{5}|$

$= \left| \dfrac{(x+1) - 5}{\sqrt{x+1} + \sqrt{5}} \right| = \left| \dfrac{x-4}{\sqrt{x+1}+\sqrt{5}} \right|$

Since $\sqrt{x+1} + \sqrt{5} \geq \sqrt{5}$

$$\Rightarrow \quad |f(x) - L| \leq \frac{|x-4|}{\sqrt{5}} < \epsilon$$

provided $\quad |x-4| < \epsilon\sqrt{5}$

$\therefore$ Choose $\delta = \epsilon\sqrt{5}$

Then $0 < |x-4| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Hence $\lim\limits_{x \to 4} \sqrt{x+1} = \sqrt{5}$.