Let $T: V \to V$ be linear. A Subspace W of V is said to be T-invariant if $x \in W \Rightarrow T(x) \in W$ (or $T(x) \in W$ for each $x \in W$)

that is, $T(W) \subseteq W$.

Q. Prove that the subspaces $\{0\}$, V, $R(T)$ and $N(T)$ are all T-invariant.

Sol^n// $\because T(0) = 0 \Rightarrow \{0\}$ is T-invariant

$\because T: V \to V$ is linear transformation

$\Rightarrow T(V) \subseteq V \Rightarrow V$ is T-invariant

$R(T) \subseteq V$ Let $R(T) = W$.

Let $x \in R(T) \Rightarrow x \in V$

$\Rightarrow T(x) \in T(V)$

$\Rightarrow T(x) \in R(T)$

$T\{R(T)\} \subseteq R(T)$.

$N(T) \subseteq V$

let $x \in N(T) \Rightarrow T(x) = 0$

$\Rightarrow T\{T(x)\} = T(0)$

$\Rightarrow T\{T(x)\} = 0$

$\Rightarrow T(x) \in N(T)$

$\therefore$ $N(T)$ is T - invariant.

<u>Def^n:</u> If W is T-invariant and $T: V \to V$ is linear. Then the restriction of T on W is the function $T_W : W \to W$ defined by

$$T_W(x) = T(x) \quad \forall\, x \in W.$$

Q. Let $T: V \to V$ be linear and W be a subspace of V.

Suppose that W is T-invariant.
Prove that $N(T_W) = N(T) \cap W$ and
$R(T_W) = T(W)$

Soln // $W \subseteq V$, $T_W(x) = T(x) \; \forall x \in W$.

let $x \in N(T_W) \Rightarrow T_W(x) = 0 \Rightarrow T(x) = 0$ for $x \in W$

$\Rightarrow x \in N(T)$ for $x \in W$

$\Rightarrow x \in N(T) \cap W$

$\therefore N(T_W) \subseteq N(T) \cap W$ ——— (1)

let $x \in N(T) \cap W \Rightarrow x \in N(T)$ & $x \in W$
$\Rightarrow T(x) = 0$ & $x \in W$

$\Rightarrow T_W(x) = 0$

$\Rightarrow x \in N(T_W)$

$\therefore N(T) \cap W \subseteq N(T_W)$ — (2)

from (1) & (2), $N(T_W) = N(T) \cap W$

Similarly, try to prove that

$$R(T_W) = T(W).$$