$\varepsilon = (1\,2)(1\,2) \rightarrow$ no. of 2-cycles $= 2$ (even)

It is an even Permutation.

Chapter 9 Normal subgroups and Factor Groups:

Normal Subgroup:

A subgroup H of a group G is called a normal subgroup of G if $aH = Ha \ \forall a \in G$

Here $aH \rightarrow$ left coset of H containing a

$Ha \rightarrow$ Right coset of H containing a.

Notation: $H \triangleleft G$

H is normal subgroup of G.

Subgroup:
H is subgroup of G
$H \leq G$

Thm Normal subgroup Test:

A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H \quad \forall x \in G$.

Proof: First, let H be a normal subgroup of G.

$\therefore \quad xH = Hx \quad \forall x \in G$

Now let $h \in H$, $x \in G$,

TS $\quad xhx^{-1} \in H$.

Now, $\quad xh \in xH$

$\Rightarrow xh \in Hx \quad (\because xH = Hx)$

$\Rightarrow \quad xhx^{-1} \in Hxx^{-1}$

$\Rightarrow \quad xhx^{-1} \in H$

$\Rightarrow \quad xHx^{-1} \subseteq H$ ($\because$ x & h are arbitrary elements of G & H respectively)

Conversely ($\Leftarrow$) Let $xHx^{-1} \subseteq H \quad \forall x \in G$.

Given that $xHx^{-1} \subseteq H$

$\therefore \Rightarrow \quad xH \subseteq Hx$ —— (1)

$A \subseteq B$
$|A| = |B|$
$\Rightarrow A = B$.

$\therefore \quad |xH| = |Hx|$

using eqn (1), $\quad xH = Hx \quad \forall x \in G$

$\therefore H$ is a normal subgroup of G.

Example ① Every subgroup of an Abelian group is normal.

$ab = ba \quad \forall \ a, b \in G$.

$H \subseteq G$, $\quad aH = Ha \quad (\because ah = ha)$

$\therefore H \triangleleft G$.

② The center $Z(G)$ of a group G is always normal in G.

③ A_3 is normal in S_3

$A_3 = \{ (1), (1\,3\,2), (1\,2\,3) \}$

$S_3 = \{ (1), (1\,2), (2\,3), (1\,3), (1\,3\,2), (1\,2\,3) \}$

$\phi A_3 = A_3 \phi \quad \forall \ \phi \in S_3$

Factor Group (or Quotient Group):

If H is a normal subgroup of G, then the set of left (or right) cosets of H in G form a group - called the factor group of G by H.

In other words: If $H \triangleleft G$, then the set $G/H = \{aH \mid a \in G\}$ is called a factor group under the operation $(aH)(bH) = abH$

Thm: If H is a normal subgroup of a group G, then the set $G/H = \{aH \mid a \in G\}$ is a group under the operation

$$(aH)(bH) = abH \quad \forall\, aH, bH \in G/H.$$

Proof: $\phi : G/H \times G/H \longrightarrow G/H$.

$f : A \to R$

$$\phi\{(aH), (bH)\} = abH$$

$f : A \to K$

well defined	one-one
$x = y$	$f(x) = f(y)$
$\Rightarrow f(x) = f(y)$	$\Rightarrow x = y$

Suppose that $aH = a'H$, $bH = b'H$

$\therefore (aH)(bH) = (a'H)(b'H)$ —— (1)

TS $abH = a'b'H$.

Now, $aH = a'H$ and $bH = b'H$

$aH = a'H$
$\Rightarrow ah = a'h'$
$\Rightarrow a' = ah(h')^{-1} = ah_1$

$\Rightarrow a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2 \in H$.

Now, $a'b'H = ah_1 b\underbrace{h_2 H}$ $\quad (aH = H \Leftrightarrow a \in H)$

$= ah_1 bH$

$= a\underbrace{h_1 H} b \quad (\because H \triangleleft G, \; bH = Hb)$

$= aHb$

$= abH$.

(1) Closedness (Binary operation):

Let $aH, bH \in G/H$.

$\Rightarrow (aH)(bH) = abH \in G/H$.

$(\because ab \in G \;\forall\; a, b \in G)$
$(\because abH$ is left coset of $H)$

② Associativity:

$aH, bH, cH \in G/H$.

$aH\{(bH)(cH)\} = aH\{bcH\} = abcH$.

and $\{(aH)(bH)\}(cH) = \{abH\}(cH) = abcH$.

Operation is Associative.

③ Identity:

$e \in G$ is the identity in G.

$eH = H$ is the identity of G/H.

$(aH)(eH) = (ae)H = aH \quad \forall \ aH \in G/H$

and $(eH)(aH) = (ea)H = aH \quad \forall \ aH \in G/H$.

④ Inverse:

$\forall a \in G, \ \exists \ a^{-1} \in G$ ($\because G$ is a group)

$\forall \ (aH) \in G/H, \quad (aH)^{-1} = a^{-1}H$.

$(aH)(a^{-1}H) = (aa^{-1})H = eH = H$

and $(a^{-1}H)(aH) = (a^{-1}a)H = eH = H$.

Thm **The G/Z Theorem:**

Let G be a group and $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then G is abelian.

Proof: $\because$ $Z(G)$ is normal in G

$\therefore$ $G/Z(G)$ is defined.

Given that $G/Z(G)$ is cyclic group.

So let $gZ(G)$ be the generator of $G/Z(G)$.

Now let $a, b \in G$

$\Rightarrow$ $aZ(G),\ bZ(G) \in G/Z(G)$

$\Rightarrow$ there exists integers m and n such that

$\{gZ(G)\}^m = aZ(G)$ and

$\{gZ(G)\}^n = bZ(G)$.

$\Rightarrow$ $g\, Z(G) = a\, Z(G)$ and $g\, Z(G) = b\, Z(G)$

$aH = a'H$.
$\Rightarrow a' = ah_1$

$\{g Z(G)\}^m = g Z(G) \cdot g Z(G) \cdots g Z(G)$
$= \underbrace{(g \cdot g \cdots g)}_{m \text{ times}} Z(G) = g^m Z(G)$

$\Rightarrow$ $a = g^m x$ and $b = g^n y$ for some $x, y \in Z(G)$.

Now
$$ab = g^m x g^n y = g^m (x g^n) y$$
$$= g^m (g^n x) y \quad (\because Z(G) \text{ is Normal in } G)$$
$$= (g^m g^n) x y$$
$$= (g^{m+n}) x y$$
$$= (g^{n+m}) x y$$
$$= (g^n g^m) x y$$
$$= (g^n g^m) y x \quad (\because x, y \in Z(G))$$
$$= g^n (g^m y) x$$
$$= g^n (y g^m) x$$
$$= (g^n y)(g^m x) = ba$$

$\therefore$ a & b are arbitrary elements of G

$\therefore ab = ba \;\forall\, a, b \in G$

$\Rightarrow G$ is abelian.

Q. Construct the Cayley table for $\mathbb{Z}/4\mathbb{Z}$.

Sol^n $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$4\mathbb{Z} = \{0, \pm 4, \pm 8, \pm 12, \ldots\}$

$4\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. (Homework)

$\because \mathbb{Z}$ is abelian $\Rightarrow 4\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$

$\Rightarrow \mathbb{Z}/4\mathbb{Z}$ is defined.

$0 + 4\mathbb{Z} = \{0, \pm 4, \pm 8, \pm 12, \ldots\} = 4\mathbb{Z}$

$1 + 4\mathbb{Z} = \{1, 5, 9, \ldots, -3, -7, -11, \ldots\}$

$2 + 4\mathbb{Z} = \{2, 6, 10, \ldots, -2, -6, -10, \ldots\}$

$3 + 4\mathbb{Z} = \{3, 7, 11, \ldots, -1, -5, -9, \ldots\}$

$4 + 4\mathbb{Z} = \{4, 8, 12, \ldots, 0, -4, -8, \ldots\} = 4\mathbb{Z}$.

The distinct left cosets of $4\mathbb{Z}$ are $4\mathbb{Z}$, $1+4\mathbb{Z}$, $2+4\mathbb{Z}$ and $3+4\mathbb{Z}$ only.

Similarly, $5+4\mathbb{Z}=1+4\mathbb{Z}$, $6+4\mathbb{Z}=2+4\mathbb{Z}$, ...

$\boxed{4\mathbb{Z}=0+4\mathbb{Z}}$

	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$

$$(aH)(bH) = abH.$$

$$(a+4\mathbb{Z}) + (b+4\mathbb{Z}) = (a+_4 b)+4\mathbb{Z}$$

Results: (G/Z theorem)

① If G/H is cyclic, where H is a subgroup of $Z(G)$, then G is abelian.

② Contrapositive of G/Z theorem

If G is not abelian, then $G/Z(G)$ is not cyclic.

Q. Construct the cayley table for G/K, where $G = D_4$, $K = \{R_0, R_{180}\}$

Sol^n

Clearly, K is a subgroup of D_4.

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$$

$$xK = Kx \quad \forall\, x \in D_4.$$ H.W.

$$R_0 K = \{R_0, R_{180}\} = K.$$

$$R_{90} K = \{R_{90}, R_{270}\}$$

$$R_{180} K = \{R_{180}, R_0\} = K$$

$$R_{270} K = \{R_{270}, R_{90}\} = R_{90} K.$$

$$HK = \{H, V\}$$

$$VK = \{V, H\} = HK.$$

$$DK = \{D, D'\}$$

$$D' K = \{D', D\} = DK.$$

there are four distinct left cosets of K in D_4
namely, K, $R_{90}K$, HK and DK.

	K	$R_{90}K$	HK	DK
K	K	$R_{90}K$	HK	DK.
$R_{90}K$	$R_{90}K$	K.	DK	HK
HK	HK	DK	K.	$R_{90}K$.
DK	DK.	HK.	$R_{90}K$.	K

$(aK)(bK)$
$= abK$.