Ch- 6 (Residue & poles)

(68) Isolated Singuler points

Singular point: A point z_0 is called a singular point if function f is fails to be analytic at z_0 but it is analytic at some point in every nbd of z_0.

Isolated singuler points: z_0 is said to be Isolated singular point of $f(z)$ if there exist a nbd of z_0 which includes no singuler point of $f(z)$ other than z_0.

i.e $\exists$ $\delta > 0$ s.t $0<|z-z_0|<\delta$ has no singuler point of $f(z)$ (i.e ~~delete~~ deleted nbd $0<|z-z_0|<\epsilon$ of z_0 throughout which f is analytic)

Example ① $f(z) = \dfrac{z+1}{z^3(z^2+1)}$

The function $f(z)$ is fails to be analytic at $z^3(z^2+1) = 0$

$\Rightarrow z = 0$ | $z = \pm i$

$z = 0, \pm i$ are singuler points of $f(z)$

Moreover, these are Isolated singular points since, $f(z)$ is analytic in deleted nbd $0<|z-z_0|<\epsilon$ where $z_0 = 0, \pm i$

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Example the function $f(z) = \frac{1}{\sin(\pi/z)}$

The $f(z) = \frac{1}{\sin(\pi/z)}$ is fails to be analytic

at $z = 0$ & $z = \frac{1}{n}$ $(n = \pm 1, \pm 2, \dots)$,

these all lying on line segment of real axis from $z = -1$ to $z = 1$

zeros of $\sin(\pi/z)$ are [$\sin(\pi/z) = 0$
$\pi/z = n\pi$, $n \in \mathbb{Z}$
$z = \frac{1}{n}$]

and $\lim_{n \to \infty} \frac{1}{n} = 0$

we get nbd of 0 contains infinite singularities.

Each singularity of form $z = \frac{1}{n}$, $n \in \mathbb{Z}$ are isolated singularity, since

{Isolated means we can draw a ball around the point & have no other singularities be that ball}

Here $z = \frac{1}{n}$, $n \in \mathbb{Z}$, we can draw sufficiently small ball s.t no other singularity of form $z = \frac{1}{m_1}$; $m_1 \in \mathbb{Z}$ ~~excepto~~ occurs in that ball

So, $z = \frac{1}{n}$; $n \in \mathbb{Z}$ are Isolated singularity

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On other hand, $z = 0$ are Not isolated singularity because every deleted ϵ nbd of $z = 0$ contains other singularity of $f(z)$ also.

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$\therefore$ $z = \frac{1}{n}$; $n \in \mathbb{Z}$ are Isolated singular point
& $z=0$ are Non-isolated singular point.

(6.9) <u>Residue</u>

When z_0 is an Isolated singular point of a function f, there is positive number R_2 s.t f is analytic at each point z for which $0 < |z - z_0| < R_2$.

We have, $f(z)$ has a Laurent series representation [by theorem on Article 60] Laurent Expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$
$$\cdots + \frac{b_n}{(z-z_0)^n} + \cdots \quad \text{—(1)}$$

where $0 < |z - z_0| < R_2$

and the coefficient a_n & b_n have certain ~~diaperees~~ integral representation (Article 60) theorem

we have,
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)^{-n+1}} \quad (n = 1, 2, \cdots)$$

where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < R_2$

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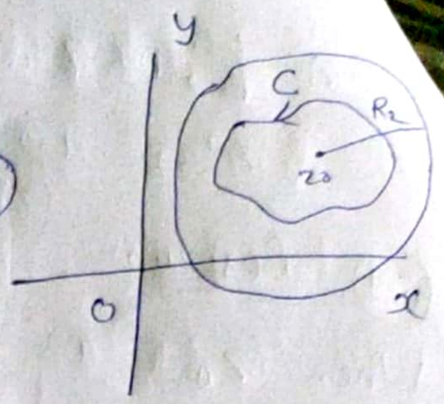
when $n=1$, this expression for b_n, becomes



$$\int_C f(z)\,dz = 2\pi i\, b_1 \quad -(2)$$

The complex number b_1, which is coefficient of $\frac{1}{(z-z_0)}$ in eqⁿ (1), Is called residue of f at Isolated singular point z_0,

i.e $b_1 = \underset{z=z_0}{\text{Res}}\, f(z)$

∴ equation (2), becomes

$$\int_C f(z)\,dz = 2\pi i \underset{z=z_0}{\text{Res}}\, f(z) \quad -(3)$$

Sometimes, B denote the residue, when function f & the z_0 prescribe above.

Example (1) consider $\int_C z^2 \sin\left(\frac{1}{z}\right) dz$

where C is positively oriented unit circle $|z|=1$

Since $z^2 \sin\frac{1}{z}$ is analytic in finite plane except at $z=0$

Using eqⁿ (3) $\int_C f(z)\,dz = 2\pi i \underset{z=z_0}{\text{Res}}\, f(z)$

To find residue, use Maclaurin series representation

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \quad |z| < \infty$$

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Now, replace $z \to \frac{1}{z}$

and then multiply by z^2

we get

$$z^2 \sin\left(\frac{1}{z}\right) = z - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}\cdot\frac{1}{z^3} - \frac{1}{7!}\,\frac{1}{z^5} + \cdots \quad \text{— (A)}$$

$$(0 < |z| < \infty)$$

Now, Residue is coefficient of $\frac{1}{z}$ in eqn (A)

Here, coefficient of $\frac{1}{z}$ is

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}.$$

Example (2) show that

$$\int_C \exp\left(\frac{1}{z^2}\right) dz = 0$$

where C is positively oriented $|z| = 1$.

Since $\frac{1}{z^2}$ is analytic everywhere except at $z = 0$

Also, $z = 0$ is Isolated singular point is in interior to C.

Use, Maclaurin series,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z| < \infty)$$

Now, replace $z \to \frac{1}{z^2}$, we get

$$\exp\left(\frac{1}{z^2}\right) = 1 + \frac{1}{1!}\cdot\frac{1}{z^2} + \frac{1}{2!}\cdot\frac{1}{z^4} + \cdots$$

$$(0 < |z| < \infty)$$

Now, Residue is coefficient of $\frac{1}{z}$

Here Coefficient of $\frac{1}{z}$ is zero

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$\therefore \int_C \exp\left(\frac{1}{z^2}\right) dz = 2\pi i \operatorname{Res}_{z=0} f(z)$

$= 2\pi i (0) = 0$

Remark :

Analyticity of a function within & on a simple closed contour C is sufficient condition for the value of Integral around C to be zero. But it is Not necessary condition.

Example ③ find $\int \frac{dz}{z(z-2)^4}$

where C is positively oriented circle $|z-2| = 1$

function $\frac{1}{z(z-2)^4}$ is Not analytic at

$z = 0, 2$

Here $z = 0$ lies outside C

But $z = 2$ lies inside C

As $|z-2| = 1$

$z = 0$

$\Rightarrow |0-2| = 2 > 1$

$\left[\begin{array}{l} |z-2| = 1 \\ z = 2 \Rightarrow |2-2| = 0 < 1 \end{array} \right]$

$\therefore$ value of $\int \frac{dz}{z(z-2)^4}$ is $2\pi i$ times residue of its integrand at $z = 2$

To determine residue, Use,

Maclaurin Series expansion

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$

Now, $\frac{1}{z(z-2)^4} = \frac{1}{(z-2)} \cdot \frac{1}{(z-2+2)}$

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$$= \frac{1}{(z-2)^4} \cdot \frac{1}{(z+2-2)}$$

$$= \frac{1}{2(z-2)^4} \cdot \frac{1}{1-\left(-\frac{(z-2)}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-4} \quad -(B) \quad (0<|z-2|<2)$$

expancing equation (B), we get coefficient of $\frac{1}{(z-2)}$

is $-\frac{1}{16}$

$$\therefore \int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(-\frac{1}{16}\right) = -\frac{\pi i}{8}.$$