Chapter-4

(4.1) Group Actions and permutation Representation

Now, apply basic theory of group action to the theory of subgroups of S_n acting on $\{1, 2, 3, \dots, n\}$ for that every element of S_n has unique cycle decomposition.

let G be a group acting on a non-empty set A.
i.e for each $g \in G$ the map

$$\sigma_g : A \to A$$

defined by $\sigma_g : a \mapsto g \cdot a$
i.e $\sigma_g(a) = g \cdot a$ is a permutation of A.

There is a homomorphism associated to an action of G on A.

$$\phi : G \to S_A$$

defined by $\phi(g) = \sigma_g$
is called permutation representation associated to given action.

Definition

① Kernel of the action is set of elements of G that act trivially on every element of A i.e $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$

② For each $a \in A$, the stabilizer of a in G is set of elements of G that fix the element a and denoted by G_a.
i.e $\{g \in G \mid g \cdot a = a\}$

(3) An action is faithful if its kernel is identity.

Remark: Kernel of an action is precisely the same as kernel of associated permutation representation. In particular, kernel is normal subgrp of G.

Examples

(1) let n be positive integer. The group $G = S_n$ acts on set $A = \{1, 2, \dots, n\}$ by

$\sigma \cdot i = \sigma(i)$ for all $i \in \{1, 2, \dots, n\}$

The permutation representation associated to this action is identity map

$$\phi : S_n \to S_n$$

This action is faithful and for each $i \in \{1, 2, \dots, n\}$ the stabilizer G_i (the subgrp of all permutations fixing i) is isomorphic to S_{n-1}

i.e to show: $G_i \approx S_{n-1}$

Define $G_i = \{\sigma \in S_n \mid \sigma(i) = i\}$ the stabilizer of element i.

i.e G_i contains the permutation that leave i fixed.

W.L.O.G suppose $i = n$

Every permutation of $f \in S_{n-1}$ can be interpreted as permutations of S_n

which stabilizer n;

$$f(1,2,3,\dots,n-1) = f(1,2,\dots,n-1,n)$$

to show! this is one to one correspondence

Consider mapping

$$f(1,2,3,\dots,n-1) \xrightarrow{\phi} f(1,2,3,\dots,n-1,n)$$

clearly ϕ is 1-1 onto (since permutations remain really the same)

Now, if $f, g \in S_{n-1}$

then $\phi(f \circ g) = f \circ g$

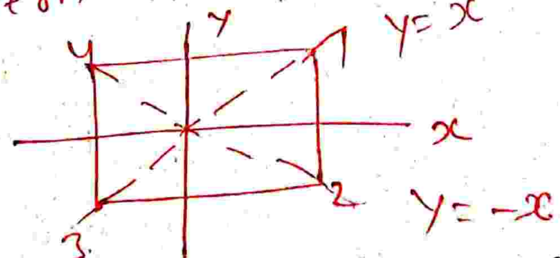
$= \phi(f)\,\phi(g)$

$\therefore$ ϕ is homomorphism

$\therefore$ $G \cong S_{n-1}$

(2) let $G = D_8$ act on set A consisting of the four vertices of a square.

Label these vertices 1, 2, 3, 4 in clockwise direction as



let r be the rotation of square clockwise by $\pi/2$ radians and let s be reflection in the line which passes through vertices 1 and 3.

Then permutations of the vertices given by r and s are

$\sigma_r = (1\,2\,3\,4)$ & $\sigma_s = (2\,4)$

we already know permutation representation is homomorphism
Now permutation of four vertices ~~of~~ to
~~square is faithful since~~

sr is $\sigma_{sr} = \sigma_s \sigma_r = (24)(1234)$

$= (14)(23)$.

The action of D_8 on four vertices of square is faithful since only the identity symmetry fixes all four vertices.

Now stabilizer of any vertex a is subgroup of D_8 of order 2 generated by the reflection about the line passing through a and center of square.

Example ③ & ④ (try as exercise).

Remark: The relation between actions and homomorphism into symmetric groups may be reversed.

For any non-empty set A and any homomorphism ϕ of group G into S_A we obtain action of G on A by defining

$$g \cdot a = \phi(g)(a) \quad \text{for all } g \in G \text{ and all } a \in A.$$

The kernel of this action is same as $\operatorname{Ker} \phi$.

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Proposition ① For any group G and non-empty set A there is bijection between the actions of G on A and homomorphism of G into S_A.

Definition: If G be group, a permutation representation of G is any homomorphism of G into the symmetric group S_A for some non-empty set A.

Group Action of G on A affords or induces the associated permutation representation of G.

Proposition ② Let G be group acting on the non-empty set A. The relation on A defined by $a \sim b$ if only if $a = g \cdot b$ for some $g \in G$ is an equivalence relation.

For each $a \in A$, the number of elements in the equivalence class containing a is $\left|\frac{G}{G_a}\right|$ the index of stabilizer of a.

(proof) To prove $\sim$ is an equivalence relation Using second axiom 2 of action

$a = 1 \cdot a$ for all $a \in A$.

i.e $a \sim a$ is reflexive relation

If $a \sim b$ then $a = g \cdot b$ for some $g \in G$

So that $g^{-1} \cdot a = g^{-1} \cdot (g \cdot b)$

$= (g^{-1} \cdot g) \cdot b$

$= 1 \cdot b$

$= b$

that is $b \sim a$, the reflexive is symmetric

Finally if $a \sim b$ & $b \sim c$
then $a = g \cdot b$ & $b = h \cdot c$ for some $g, h \in G$

so, $a = g \cdot b$
$= g \cdot (h \cdot c)$
$= (gh) \cdot c$

$\Rightarrow a \sim c$, relation is transitive.

For any $a \in A$, equivalence class of a is
~~$C_a = \{x \in A ; x \sim a\}$~~
~~$= x \in A ; x =$~~

$$C_a = \{g \cdot a \mid g \in G\}$$

suppose $b = g \cdot a \in C_a$
let $\frac{G}{G_a}$ be set contains all left cosets of G_a in G.

$\phi : C_a \to \frac{G}{G_a}$ such that
$\phi(g \cdot a) = g G_a$

ϕ is well defined & 1-1

Let $g_1 \cdot a = g_2 \cdot a$

$\Leftrightarrow g_1^{-1} \cdot (g_1 \cdot a) = g_1^{-1} \cdot (g_2 \cdot a)$

$\Leftrightarrow (g_1^{-1} \cdot g_1) \cdot a = (g_1^{-1} \cdot g_2) \cdot a$

$\Leftrightarrow e \cdot a = (g_1^{-1} \cdot g_2) \cdot a$

$\Leftrightarrow a = (g_1^{-1} \cdot g_2) \cdot a$ (by definition of stabilizer)

$\Leftrightarrow g_1^{-1} \cdot g_2 \in G_a$ ($aH = bH \Leftrightarrow a^{-1}b \in H$)

$\Leftrightarrow g_1 G_a = g_2 G_a$

$\Leftrightarrow \phi(g_1 \cdot a) = \phi(g_2 \cdot a)$

ϕ is onto

since for any $g \in G$, the element $g \cdot a$ is an elt of C_a

($\therefore$ No. of elts in any equivalence class is equal to the number of index of G_a in G)

Result: $\phi : G_a \to \frac{G}{G_a}$

~~$\phi(G_a) = \phi\left(\frac{G}{G_a}\right) = 0$~~

$O(G_a) = O\left(\frac{G}{G_a}\right) = \frac{O(G)}{O(G_a)}$

$\Rightarrow O(G) = O(G_a) \cdot O(G_a)$

This G_a is called orbit

Definition: Let G be group acting on non-empty set A.

1. The equivalence class $\{g \cdot a \mid g \in G\}$ is called orbit of G containing a.
2. The action of G on A is called transitive if there is only one orbit

i.e given any two elements $a, b \in A$
there is some $g \in G$ such that $a = g \cdot b$.

Examples

Let G be group acting on set A

① let G be group and if G acts trivially on A then, orbit of are elts of A

(sol^n) let $*$ be trivial action,

$g * a = a \quad \forall a$

$\therefore$ orbit of a, $Ga = \{g * a \mid g \in G\}$

$= \{a\}$

(II part) if $* : G \times A \to A$ be trivial action, then the action is transitive iff $|A| = 1$

(proof) Let $*$ be transitive (by definition ②)

$\exists$ only one orbit say Ga

under trivial action we know that any orbit is of type $\{a\}$

thus $A = \{a\}$

$|A| = 1$

Conversly, let $|A| = 1$

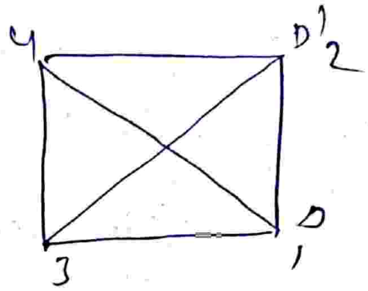
let $A = \{a\}$

$\therefore$ there is only one orbit (As orbit is subset of A)

Example ②

Example: The group D_8 acts transitively on four vertices of square and stabilizer of any vertex is subgrp of order 2 (index 4) generated by reflection about the line of symmetry passing through that point.

(Sol^n) D_8 acts on $A = \{1, 2, 3, 4\}$



$D_8 = \{R_0, R_{90}, R_{270}, R_{180}, H, V, D, D'\}$

orbit of 1

$= \{g \cdot 1 \ \forall g \in D_8\}$

$= \{R_0 \cdot 1, R_{90} \cdot 1, R_{180} \cdot 1, R_{270} \cdot 1, H \cdot 1, V \cdot 1, D \cdot 1, D' \cdot 1\}$

$= \{R_0(1), R_{90}(1), R_{180}(1), R_{270}(1), H(1), V(1),$
$D(1), D'(1)\}$

$= \{1, 2, 3, 4\} = A$

orbit of $2 = \{R_0 \cdot 2, R_{90} \cdot 2, R_{180} \cdot 2, \dots \}$

$= \{2, 3, 4, 1\} = A$

Similarly others

So, there is only one orbit $= A$

$\therefore D_8$ acts transitively on $A = \{1, 2, 3, 4\}$

Stabilizer: $G_a = \{g \in G \mid g \cdot a = a\}$

$G_1 = \{g \in D_8 \mid g \cdot 1 = 1\}$

$G_1 = \{g \in D_8 \mid g(1) = 1\}$

$= \{R_0, D\}$

G_2 = Stabilizer of 2 = $\{R_0, D'\}$

$G_3 = \{R_0, D\}$

$G_4 = \{R_0 D'\}$

$$\left|\frac{G}{G_a}\right| = \frac{|G|}{|G_a|} = \frac{8}{2} = 4 \quad (\text{index of stabilizer of } a)$$

i.e the number of elements in equivalence class containing a is 4.

Cycle decomposition:

Every element of the symmetric group S_n has the unique cycle decomposition.

let $A = \{1, 2, \ldots, n\}$. Let σ be an element of S_n and let $G = \langle\sigma\rangle$. Then $\langle\sigma\rangle$ acts on A

By using proposition ②

In other words

a group G acting on a set A partitions A into disjoint equivalence classes under the action of G.

we gets, its partitions $\{1, 2, \ldots, n\}$ into a unique set of (disjoint) orbits.

let O be one of these orbits and let $x \in O$

(as proof of proposition 2) applied to $A = O$

there is 1-1 onto mapping from left cosets of G_x in G and elements of O

given by (mapping: $O \to \frac{G}{G_x}$)

$$\sigma^i x \mapsto \sigma^i G_x$$

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Since G is cyclic group, $G_x \trianglelefteq G$ and $\frac{G}{G_x}$ is cyclic of order d, where d is smallest positive integer for which $\sigma^d \in G_x$

Also, $d = \left(\frac{G}{G_x}\right) = |O|$

thus distinct cosets of G_x in G are
$1 \cdot G_x, \sigma G_x, \sigma^2 G_x, \ldots, \sigma^{d-1} G_x$

Thus shows that the distinct elements of O are
$x, \sigma(x), \sigma^2(x), \ldots, \sigma^{d-1}(x)$ in this manner

Ordering the elements of O shows the σ cycles the elements of O.

Since an orbit of size d, σ acts on d-cycle. Therefore existence of a cycle decomposition for each $\sigma \in S_n$.

The orbit of $\langle\sigma\rangle$ are uniquely determined by σ. Choosing $\sigma^i(x)$ instead of x as the initial representative produces the elements of O in the order

$\sigma^i(x), \sigma^{i+1}(x), \ldots \sigma^{d-1}(x),$

$x, \sigma(x), \ldots, \sigma^{i-1}(x)$.

which is cycle permutation of the list.

$\therefore$ cycle decomposition above is unique up to rearrangements of the cycles and upto cyclic permutation of integers within each cycle.

(9) Let G be group and let A be a non-empty set
① Let G acts on the set A. Prove that
if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$.
then $G_b = g G_a g^{-1}$. (G_a is stabilizer of a)
Deduce that if G acts transitively on A then the kernel of action $\bigcap_{g \in G} g G_a g^{-1}$.

(Sol^n) to show! $g G_a g^{-1} = G_a$

Let an elt of $g G_a g^{-1}$

it is of form $g h g^{-1}$, where $h \in G_a$

then $(g h g^{-1}) \cdot b = (g h g^{-1}) \cdot (g \cdot a)$

$= (g h)(g^{-1} \cdot g) \cdot a$

$= (g h) \cdot a$

$= g \cdot (h \cdot a)$

$= g \cdot a$ (since $h \in G_a$)

$= b$

(As $h \in G_a \Rightarrow h = g \cdot a$ for some $g \in G$)

thus $g G_a g^{-1} \subseteq G_b$

Conversely, Let $x \in G_a$ & $b = g \cdot a$

$\Rightarrow x \cdot b = b$

claim: $g^{-1} x g \in G_a$

Consider, $(g^{-1} x g) \cdot a = (g^{-1} x) \cdot (g \cdot a)$

$= (g^{-1} x) \cdot b$

$= g^{-1} (x b)$

$\Rightarrow g^{-1} x g \in G_a \quad = g^{-1} (b)$

$g^{-1} x g \in G_a$

$\Rightarrow x \in g G_a g^{-1}$

$\Rightarrow G_b \subseteq g G_a g^{-1}$

$\therefore G_b = g G_a g^{-1}$

Suppose that G acts transitively on A (only one orbit)
this implies that any element $a \in A$ can be
transformed to any other element $b \in A$
by acting with some element of group G

i.e $b = g \cdot a$

Since kernel is by definition the set of all $g \in G$
that stabilize all elements,
thus it's intersection of all stabilizer

$$\bigcap_{b \in A} G_b$$

Since the group acts transitively, kernel becomes
this intersection $g \in G$ & so,

$$\bigcap_{b \in A} G_b = \bigcap_{g \in G} G_{g \cdot a} = \bigcap_{g \in G} g G_a g^{-1}$$

(2) let G be a permutation group on set A. Prove that
let $\sigma \in G$ and let $a \in A$. Deduce that if
$\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. on A then
G acts transitivity

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1$$

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(Solution) G be given to be permutation group on set A. By above (1) problem we have if $\sigma \in G \Rightarrow \sigma G_a \sigma^{-1} = G_{\sigma(a)}$

If assume the action is transitive then kernel $= \bigcap_{\sigma \in G} \sigma G_a \sigma^{-1}$

As transitive, so, kernel is 1

$\Rightarrow \bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1$

(3) Assume that G is abelian, transitive subgroup of S_A. Show that $\sigma(a) \neq a$ for all $\sigma \in G - \{1\}$ and all $a \in A$ then by deduce $|G| = |A|$

(Soln) Since, G acts transitivity above question (1) & (2) $\sigma G_a \sigma^{-1} = \{1\}$

$\ker = \bigcap_{\sigma \in G}$

& $\ker = \{\sigma \in G \mid \sigma . a = a\}$

if $\sigma \neq 1$ then $\sigma(a) \neq a \quad \forall a \in A$

$\Rightarrow \sigma(a) \neq a \quad \forall \sigma \in G - \{1\}$ & $\forall a \in A$.

show) $|G| = |A|$ (try as exercise)

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