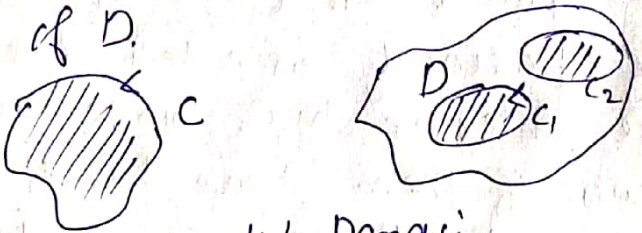
(70) Cauchy's Residue Theorem

Simply Connected Domain

Definition Simply Connected Domain
A simply connected domain D, is a domain such that every simple closed contour within it encloses only points of D.

Example (1)



C is simple connected Domain
D is simple connected Domain, since every simple closed contour, encloses only points of D

Remark: It can be said (not in general) Domain without hole is simply connected domain

Example (2)

it is Not simply connected domain, as it has hole.

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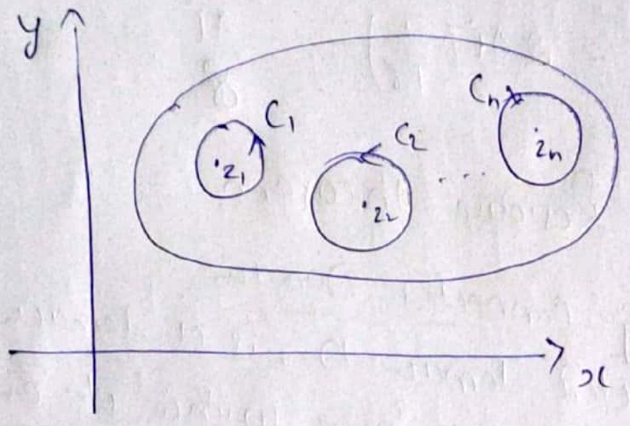
Cauchy's Residue theorem

Theorem: let C be simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)$$



(proof) let the points z_k $(k = 1, 2, \ldots n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. the circle C_k, together with simple closed contour C, form the boundary of closed region throughout which f is analytic & whose interior is multiply connected domain consisting of points inside C and exterior to each C_k.

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Multiply Connected Domain : A domain that is not simply connected is said to be multiply connected.

Using theorem (Article 49)

Suppose that

(a) C is a simple closed contour, described in the counterclockwise direction

(b) C_k $(k=1,2,\dots,n)$ are simple closed contours interior to C, all described in clockwise direction and whose interiors have no points in common

If a function f is analytic on all of those contours and throughout the multiply connected domain consisting of points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n}\int_{C_k} f(z)\,dz = 0$$

we get, (using above theorem)

$$\int_C f(z)\,dz - \sum_{k=1}^{n}\int_{C_k} f(z)\,dz = 0 \quad \text{(C is positively oriented)}$$

using, (Article 69) Residue, we have

$$\int_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad (k=1,2,\dots n)$$

∴ we have

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \quad \text{(Hence Proved)}$$

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Example $\int_C \frac{5z-2}{z(z-1)}\,dz$

where C is circle $|z|=2$, counterclockwise

$f(z) = \frac{5z-2}{z(z-1)}$ has isolated singularities

$z=0$ & $z=1$, both are interior to C

$\left[\begin{array}{l} |z|=2 \\ |0|<2, \ |1|<2 \end{array} \right]$

Using, $\frac{1}{1-z} = 1+z+z^2+\cdots \quad (|z|<1)$

Now, $0<|z|=2$

$\Rightarrow 0<|z|<1<2$

Since, $\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{(-1)}{1-z}$

$= \left(5-\frac{2}{z}\right)\left(-1-z-z^2 \cdots\right)$

we get coefficient of $\frac{1}{z}$ in above is 2

$\therefore$ residue B_1 at $z=0$ is 2

Also, Since

$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)}$

$= 5 + \frac{3}{z-1}\left(1-(z-1)+(z-1)^2 - \cdots\right)$

when $0<|z-1|<1$

we get coefficient of $\frac{1}{z-1}$ is 3

$\therefore B_2 = 3$

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$$\int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) + 2\pi i \operatorname*{Res}_{z=1} f(z)$$

$$= 2\pi i\,(B_1 + B_2)$$

$$= 2\pi i\,(2+3)$$

$$= 10\pi i.$$

(7) Residue at Infinity

Theorem: If a function f is analytic everywhere in the finite plane except for a finite ~~everywhere~~ number of singular points interior to a positively oriented simple closed contour C

then,
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

(proof) suppose function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C.

let R_1 denote a positive number which is large enough that C lies inside the circle $|z| = R_1$.

The function f is analytic throughout the domain $R_1 < |z| < \infty$, the point at infinity is then isolated singular point of f.

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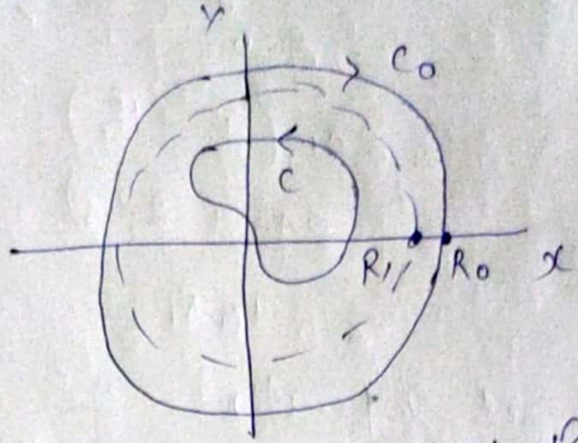

Now, let C_0 denote a circle $|z| = R_0$, oriented in the clockwise direction, where $R_0 > R_1$. Residue of f at infinity is given by

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z) \quad - (1)$$

we have to find,

$$\operatorname*{Res}_{z=\infty} f(z)$$

Using, principle of deformation of paths (see Article 49, corollary)

Since f is analytic throughout the closed region bounded by C & C_0, the principle of deformation of path, we get

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz \quad -(*)$$

Using (1) & $(*)$, we get

$$\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) \quad - (2)$$

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using, Laurent series,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1 < |z| < \infty) \quad -(3)$$

where

$$c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)\,dz}{z^{n+1}} \quad (n = 0, \pm 1, \pm 2, \cdots) \quad -(4)$$

Replacing z by $1/z$ in eqn (3) and then multiplying through the result by $\frac{1}{z^2}$ we get

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad \left(0 < |z| < \frac{1}{R_1}\right) \quad -(A)$$

and put $n = -1$ in eqn (4), we have

$$c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z)\,dz \quad -(5)$$

and using (A), we have, for $n = -1$

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = c_{-1} \quad -(6) \quad \left(\text{i.e. coeff of } \frac{1}{z}\right)$$

using (5) & (6) & (A), we get

$$\int_{C_0} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \quad -(7)$$

$\therefore$ using (1) & (7), we have

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \quad -(8)$$

Hence, using (2) & (8) equation we result

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

(Hence Proved)

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Example find $\int_C f(z)\,dz$, where $f(z) = \frac{5z-2}{z(z-1)}$

and around the circle C; $|z| = 2$, described counterwise.

(Solution) $f(z) = \frac{5z-2}{z(z-1)}$ has singular point at $z=0$ & $z=1$, which are finite number of singular points,

$\therefore$ Using above theorem

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Since, $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z}$

$$= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \cdots)$$

$$= \frac{5}{z} + 3 + 2z + \cdots$$

coeff. of $\frac{1}{z}$ is here 5

$\therefore$ desired residue is 5

Using above theorem, we have

$$\int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$= 2\pi i (5)$$

$$= 10\pi i.$$

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Excercise

(Q-1) Find the residue at $z=0$ of the function

(a) $\frac{1}{z+z^2}$ (b) $z\cos(\frac{1}{z})$

(Sol^n) $\frac{1}{z+z^2} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z}(1-z+z^2-z^3+\cdots)$

$(0<|z|<1)$

$= \frac{1}{z} - 1 + z - z^2$

The residue at $z=0$ is coefficient of $\frac{1}{z}$ is 1

(b) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

$z\cos(\frac{1}{z}) = z\left(1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots\right)$

$= z - \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^3}$

The residue at $z=0$, is coefficient of $\frac{1}{z}$ is $-\frac{1}{2}$

(c) $\frac{z-\sin z}{z}$ (try as excercise)

(Q-2) Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z|=3$ in positive sense!

(a) $\frac{\exp(-z)}{z^2}$ (b) $\frac{z+1}{z^2-2z}$

(Sol^n) $\int_{C:|z|=3} \frac{\exp(-z)}{z^2}$

$\frac{\exp(-z)}{z^2}$ has singularity at $z=0$

$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)$

$= \frac{1}{z^2} - \frac{1}{1!}\frac{1}{z} + \frac{1}{2!} \left(-\frac{z}{3!}\right) + \cdots$ $(0<|z|<\infty)$

Residue at $z=0$ is -1

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$\therefore$ By Cauchy's residue theorem, we have

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i$$

(b) $\displaystyle\int_{C:|z|=3} \frac{z+1}{z^2-2z}\,dz$ ($z=0, z=2$ both lies inside $|z|=3$)

Since, $\left(\frac{z+1}{z(z-2)}\right)$

$\frac{z+1}{z^2-2z}$ has singularity at $z=0$ & $z=2$

consider $\frac{z+1}{z(z-2)} = \frac{z+1}{z}\cdot\frac{1}{z-2}$

$= \left(1+\frac{1}{z}\right)\cdot -\frac{1}{2}\left(\frac{1}{1-z/2}\right)$ [Find coeff. of $\frac{1}{z}$]

$= -\frac{1}{2}\left(1+\frac{1}{z}\right)\cdot\left(\frac{1}{1-z/2}\right)$

$= \left(-\frac{1}{2}-\frac{1}{2}\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right)$

$=$ coeff. of $\frac{1}{z}$ is $\left(-\frac{1}{2}\right)$

$\therefore \operatorname{Res}_{z=0} \frac{z+1}{z^2-2z}$ is $-\frac{1}{2}$

Find residue at $z=2$

i.e coeff. of $\frac{1}{z-2}$, (find)

$\frac{z+1}{z(z-2)} = \frac{1}{2}\left(1+\frac{3}{z-2}\right)\cdot\frac{1}{1+(z-2)/2}$

$=$ coeff. of $\frac{1}{z-2}$ is $\frac{3}{2}$

$\therefore$ By Cauchy Residue theorem,

$$\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right)$$

$$= 2\pi i$$

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3) C is positively oriented circle $|z|=2$
find $\int \frac{z^5}{1-z^3}$

(try as excercise)

(Q-4) Let C denote the circle $|z|=1$, taken counterclockwise,

(a) Using maclaurin series for e^z, write $\int_C \exp(z+\frac{1}{2})dz = \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n \exp(\frac{1}{2})dz$

(Solution) The maclaurin series $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ $(|z|<\infty)$

we have

$$\int_C \exp(z+\tfrac{1}{2})dz = \int_C e^z e^{1/2} dz$$

$$= \int_C e^{1/2}\sum_{n=0}^{\infty}\frac{z^n}{n!} dz$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n \exp(\tfrac{1}{2})dz$$

(b) Exercise

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