9·17. Oscillations of Hanging Chain

Consider a uniform chain or a flexible string hanging from a rigid support under the action of gravity. Let the chain undergo small oscillations in a plane. Let free end of the chain be taken as reference level for distances. Let P be any point having coordinates (x, y), x being distance of chain from free end and y being small deviation.

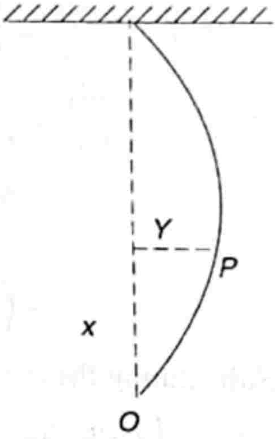

Fig. 9·9

If initially at $t = 0$, the chain is given a displacement $y_0(x)$, *i.e.*,

at $t = 0$, $y = y_0(x)$ ... (1)

then at time t, we may write

$y = y(x, t)$ ... (2)

In this case, the tension T in the chain is variable, therefore we have from equation (4) of section 9·13 :

$$m \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T \frac{\partial y}{\partial x}\right) \quad \text{... (3)}$$

where m is the mass per unit length of the chain.

In this case $T = (mx) \cdot g$

$$\text{Hence } m \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(mxg \cdot \frac{\partial y}{\partial x}\right)$$

or
$$m \frac{\partial^2 y}{\partial t^2} = mg\left(x \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\right)$$

$$\text{This gives } x \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} = \frac{1}{g} \frac{\partial^2 y}{\partial t^2} \quad \text{... (4)}$$

This equation represents the differential equations for the vibrations of the hanging chain.

By the method of separation of variables, let its solution be

$$y(x, t) = e^{i\omega t} \, v(x)$$

$$\frac{\partial y}{\partial x} = e^{i\omega t} \frac{\partial v}{\partial x}, \quad \frac{\partial^2 y}{\partial x^2} = e^{i\omega t} \frac{\partial^2 v}{\partial x^2}$$

$$\text{Also } \frac{\partial y}{\partial t} = i\omega \, e^{i\omega t} \, v(x), \quad \frac{\partial^2 y}{\partial t^2} = -\omega^2 e^{i\omega t} \, v(x)$$

Substituting these values in (4), we get

$$x \, e^{i\omega t} \frac{\partial^2 v}{\partial x^2} + e^{i\omega t} \frac{\partial v}{\partial x} = -\frac{\omega^2}{g} v \, e^{i\omega t}$$

Cancelling the common factor $e^{i\omega t}$, we get

$$x \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + \frac{\omega^2}{g} v = 0 \quad \text{... (6)}$$

This equation may be reduced to Bessel's equations by change of variable on putting

$$z^2 = \frac{4\omega^2}{g} x \quad \text{or} \quad z = \sqrt{\frac{4\omega^2}{g}} \, x^{1/2} \quad \text{... (7)}$$

$$\frac{\partial z}{\partial x} = \sqrt{\frac{4\omega^2}{g}} \; \frac{1}{2} x^{-1/2} = \frac{4\omega^2}{g} \cdot \frac{1}{2z}$$

So that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{4\omega^2}{g} \cdot \frac{1}{2z}\frac{\partial v}{\partial z}$

and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{4\omega^2}{g} \cdot \frac{1}{2z}\frac{\partial v}{\partial z}\right)$

$$= \frac{4\omega^2}{g} \cdot \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial v}{\partial z}\right)\frac{\partial z}{\partial x} = \frac{4\omega^2}{g} \cdot \frac{1}{2}\left[\frac{1}{z}\frac{\partial^2 v}{\partial z^2} - \frac{1}{z^2}\frac{\partial v}{\partial z}\right] \cdot \frac{4\omega^2}{g} \cdot \frac{1}{2z}$$

$$= \left(\frac{4\omega^2}{g}\right)^2 \cdot \frac{1}{4z^2}\frac{\partial^2 v}{\partial z^2} - \left(\frac{4\omega^2}{g}\right)^2 \cdot \frac{1}{4}\frac{1}{z^3}\frac{\partial v}{\partial z}$$

Substituting these values in (6), we get

$$\frac{g}{4\omega^2}z^2\left\{\left(\frac{4\omega^2}{g}\right)^2\frac{1}{4z^2}\frac{\partial^2 v}{\partial z^2} - \left(\frac{4\omega^2}{g}\right)^2\frac{1}{4z^3}\frac{\partial v}{\partial z}\right\} + \frac{4\omega^2}{g} \cdot \frac{1}{2z}\frac{\partial v}{\partial z} + \frac{\omega^2}{g}v = 0$$

or $$z^2\frac{\partial^2 v}{\partial z^2} + z\frac{\partial v}{\partial z} + z^2 v = 0 \qquad \text{... (8)}$$

This is Bessel's equation, whose general solution is given by

$$v = A\,J_0(z) + BY_0(z) \qquad \text{... (9)}$$

where $J_0(z)$ and $Y_0(z)$ are the Bessel's and Neumann's functions of order zero

At $x = 0$, $z = 0$ and $[Y_0(z)]_{z\to 0} \to \infty$

Therefore if displacement of chain y remains finite at $x = 0$, we must have $B = 0$. Then equation (9) would reduce to

$$v = A\,J_0(z) = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) \qquad \text{... (10)}$$

So far ω is undetermined. In order to find ω, we use the boundary conditions

$v = 0$ at $x = s$

Equation (10), then yields

$$0 = AJ_0\left(2\omega\sqrt{\frac{s}{g}}\right)$$

For a non-trivial solution $A \neq 0$, therefore $J_0\left(2\omega\sqrt{\frac{s}{g}}\right) = 0$ ... (11)

If we let $\mu = 2\omega\sqrt{\frac{s}{g}}$ ... (12)

Equation (11) becomes $J_0(\mu) = 0$ ... (13)

From the table of Bessel's function equation (13) is satisfied for values of μ given by 2·405, 5·520, 8·654, 11·792, etc.

Then the possible values of $\omega\left[= \frac{\mu}{2}\sqrt{\frac{g}{s}}\right]$ from (12) are given by

$$\omega_1 = \frac{2{\cdot}405}{2}\sqrt{\frac{g}{s}},\ \omega_2 = \frac{5{\cdot}520}{2}\sqrt{\frac{g}{s}},\ \omega_3 = \frac{8{\cdot}654}{2}\sqrt{\frac{g}{s}} \text{ etc.} \qquad \text{... (14)}$$

These correspond to a characteristic function v given by (10) for each value of ω. If v_n corresponds to ω_n, we must have

$$v_n = A_n J_0\left(2\omega_n\sqrt{\frac{x}{g}}\right) \qquad \text{... (15)}$$

Therefore the most general solution of (5) of equation (4) may be expressed as

$$y(x, t) = \sum_{n=1}^{\infty} J_0 \left(2\omega_n \sqrt{\frac{x}{g}}\right) (A_n \cos \omega_n t + B_n \sin \omega_n t) \quad \text{... (16)}$$

where A_n and B_n are arbitrary constants to be determined by the boundary conditions.

In this case the initial condition is that at $t = 0, \frac{\partial y}{\partial t} = 0$

This leads to condition $B_n = 0$

Then equations (16) at $t = 0$, becomes

$$y_0(x) = \sum_{n=1}^{\infty} A_n J_0 \left(2\omega_n \sqrt{\frac{x}{g}}\right) \quad \text{... (17)}$$

That is arbitrary displacement $y_0(x)$ may be expanded into a series of Bessel's functions of zeroth order.

Now substituting $\sqrt{\frac{x}{s}} = z$ ... (18)

equation (17) becomes $y_0(x) = \sum_{n=1}^{\infty} A_n J_0 (\mu_n z)$ ... (19)

To find coefficients A_n, we multiply (19) with $z J_0 (\mu_k z)$ and integrate between limits 0 and 1, we get

$$\int_0^l z y_0 (sz^2) J_0 (\mu_k z)\, dz = \sum A_n \int_0^l z J_0 (\mu_n z) J_0 (\mu_k z)\, dz.$$

Using orthogonal property of Bessel's functions, we get

$$\int_0^l z \cdot y_0 (sz^2) J_0 (\mu_k z)\, dz = A_k \cdot \frac{1}{2} \cdot J_1^2 (\mu_k)$$

$$\therefore \quad A_k = \frac{2}{J_1^2 (\mu_k)} \int_0^l z y_0 (sz^2) J_0 (\mu_k z)\, dz$$

Setting $k = n$, we get

$$A_n = \frac{2}{J_1^2 (\mu_n)} \int_0^l z y_0 (sz^2) J_0 (\mu_n z)\, dz \quad \text{... (20)}$$

Equations (19) and (20) determine the possible modes of oscillations of hanging chain.

9·18. Vibrations of a Rectangular Membrane

In order to consider the vibrations of a membrane, let us assume :

1. The membrane is perfectly flexible.

2. The density of membrane is uniform throughout.

3. When set in vibrations the displacement is in a direction perpendicular to membrane and is very small from equilibrium position.

Let σ be the mass per unit area of the membrane and T the normal force per unit length. If the membrane is perfectly flexible, the tension is distributed evenly throughout its area *i.e.* material on opposite sides of any line segment dx is pulled apart with a force of $T \cdot dx$.

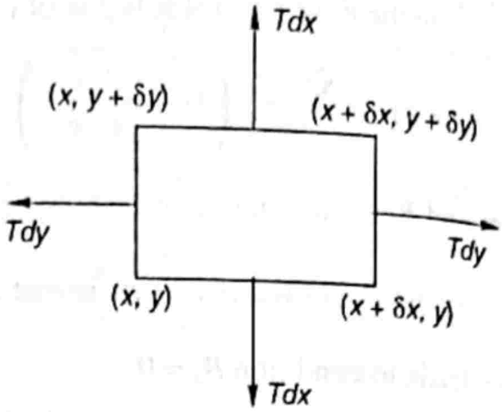

Fig. 9·10

Consider an elementary section $dx\,dy$ of the membrane in the xy plane. Let u be the displacement of the membrane from its equilibrium position. This displacement is the function of time and position of the point on the membrane. If this element of the membrane is displaced through δu in a direction perpendicular to xy plane, then the force acting in this direction is $T\,\delta u$ at length δx. Therefore the force acting per unit length is

$$T\frac{\delta u}{\delta x} = T\frac{\partial u}{\partial x} \text{ (in the limit)}$$

Therefore the force in perpendicular direction at edge $(x + \delta x)$ is

$$T\frac{\partial u}{\partial x}\bigg|_{x+\delta x} \delta y$$

While the force acting at edge x is

$$-T\frac{\partial u}{\partial x}\bigg|_{x} \delta y$$

∴ Net force normal to surface of membrane due to above forces at edges x and $x + \delta x$ is

$$T\,\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = T\frac{\partial^2 u}{\partial x^2}\delta x\,\delta y \qquad \text{... (1)}$$

Similarly, the net normal force at edges y and $y + \delta y$ is

$$T\,\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}\right] = T\frac{\partial^2 u}{\partial y^2}\delta x\,\delta y \qquad \text{... (2)}$$

Therefore the net force on the element $= T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x \cdot \delta y$ .... (3)

From Newton's second law of motion this must be equal to mass $(\sigma\,\delta x\,\delta y)$ of the element multiplied by its acceleration *i.e.*,

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x\,\delta y = (\sigma\,\delta x\,\delta y)\frac{\partial^2 u}{\partial t^2}$$

Cancelling out the common factor $\delta x\,\delta y$ on both sides, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{... (4)}$$

where $v = \sqrt{\frac{T}{\sigma}}$ is the velocity of the wave

Equation (4) represents (two-dimensional) *wave equation for the membrane.*

Solution of Wave Equation (Method of Separation of Variables)

As u is a function of x, y and t, therefore by the method of separation of variables, we may write

$$u(x, y, t) = X(x)\,Y(y)\,\tau(t) = \phi(x, y)\,\tau(t) \qquad \text{... (5)}$$

where $\phi(x, y) = X(x) Y(y)$ is the function of x and y and $\tau(t)$ is the function of t only

Substituting (5) in (4) and dividing throughout by $\phi\tau$; we get

$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\phi}\frac{\partial^2\phi}{\partial y^2} = \frac{1}{v^2\tau}\frac{\partial^2\tau}{\partial t^2} \quad \text{... (6)}$$

In this equation left hand side is the function of x and y ; while right hand side is the function of t only ; therefore each side must be equal to the same constant $\left(-\frac{\omega^2}{v^2}\right)$ say ; then we have

$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\phi}\frac{\partial^2\phi}{\partial y^2} = -\frac{\omega^2}{v^2} \quad \text{... (7)}$$

and

$$\frac{1}{v^2\tau}\frac{\partial^2\tau}{\partial t^2} = -\frac{\omega^2}{v^2} \quad \text{or} \quad \frac{\partial^2\tau}{\partial t^2} + \omega^2\tau = 0 \quad \text{... (8)}$$

The solution of equation (8) may be expressed as

$$\tau = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \text{ or } \tau = A_1 \cos\omega t + A_2 \sin\omega t \quad \text{... (9)}$$

where A_1 and A_2 are arbitrary constants.

Further, equation (7) may be expressed as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\omega^2}{v^2}\phi = 0 \quad \text{... (10)}$$

This is *ordinary Helmholtz equation* in two variables x and y and its solution may easily be obtained by method of separation of variables or by Green's function techniques.

If a and b are sides of membrane, fastened at the edges $x = 0$, $x = a$, $y = 0$ and $y = b$; then the boundary conditions are

$$\left.\begin{aligned} u &= 0 \quad \text{at} \quad x = 0 \\ u &= 0 \quad \text{at} \quad x = a \\ u &= 0 \quad \text{at} \quad y = 0 \\ u &= 0 \quad \text{at} \quad y = b \end{aligned}\right\} \quad \text{... (11)}$$

Further, if we substitute $\phi = X(x) Y(y)$ in (10) and divide by XY ; we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{\omega^2}{v^2} = 0$$

or

$$\frac{1}{X}\frac{\partial^2 Y}{\partial x^2} + \frac{\omega^2}{v^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad \text{... (12)}$$

In this equation left hand side is a function of x only ; while right hand is a function of y only ; therefore each side must be equal to the same constant (k^2) say

i.e.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{\omega^2}{v^2} = k^2 \quad \text{or} \quad \frac{\partial^2 X}{\partial x^2} + \left[\frac{\omega^2}{v^2} - k^2\right] X = 0 \quad \text{... (13)}$$

or

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \quad \text{or} \quad \frac{\partial^2 Y}{\partial y^2} + k^2 Y = 0 \quad \text{... (14)}$$

The solution of equation (13) and (14) may be expressed as

$$X = B \sin\left[\sqrt{\left(\frac{\omega^2}{v^2} - k^2\right)}\, x + \delta_x\right] \quad \text{... (15)}$$

$$Y = C \sin(ky + \delta_y) \quad \text{... (16)}$$

where B, C, δ_x and δ_y are arbitrary constants.

According to *boundary condition* $u = 0$ at $x = 0$ and $x = a$;

we have from (15), $\delta_x = 0$ and $a\sqrt{\left(\frac{\omega^2}{v^2} - k^2\right)} = m\pi$; $m = 1, 2, 3$ ... (17)

$$\therefore \quad X_m = B \sin \frac{m\pi x}{a}$$

and general solution for X is $X = \sum_m^{\infty} B_m \sin \frac{m\pi x}{a}$ ... (18)

Also according to boundary condition $u = 0$ at

$y = 0$ and b ; equation (16) gives

$\delta_y = 0$ and $kb = n\pi$; $n = 1, 2, 3...$ ... (19)

Hence $\quad Y_n = C_n \sin \frac{n\pi y}{b}$

and the general solution for Y is

$$Y = \sum_n Y_n = \sum_n C_n \sin \frac{n\pi y}{b} \quad \text{... (20)}$$

The value of ω is given by equation (17) *i.e.*,

$$\left(\frac{\omega^2}{v^2} - k^2\right) \cdot a^2 = m^2\pi^2 \text{ or } \frac{\omega^2}{v^2} = \frac{m^2\pi^2}{a^2} + k^2$$

Substituting value of k from (19) ; we get

$$\frac{\omega^2}{v^2} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

or $$\omega = v\pi \sqrt{\left\{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\right\}} = \omega_{mn} \text{ (say)} \quad \text{... (21)}$$

This equation represents the *possible angular frequencies.*

Hence the *complete solution of vibrating rectangular membrane* is given by

$$u = X(x)\, Y(y)\, \tau(t)$$

$$= \sum_m B_m \sin \frac{m\pi x}{a} \sum_n C_n \frac{n\pi y}{b} \left\{A_1 \cos \omega_{mn} t + A_2 \sin \omega_{mn} t\right\}$$

$$= \sum_m \sum_n [A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \text{... (22)}$$

where A_{mn} and B_{mn} are new arbitrary constants which may be determined from initial conditions of displacement and velocity.

Let at $t = 0$, displacement $u(x, y, t) = u_0$ ... (23)

and velocity $\dot{u}(x, y, t) = \left(\frac{du}{dt}\right)_{t=0} = \dot{u}_0$ ... (24)

Using (23) *i.e.*, at $t = 0$, $u = u_0$; equ. (22) gives

$$u_0 = \sum_m \sum_n A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \text{... (25)}$$

Multiplying both sides of the equation with $\sin \frac{m\pi x}{a}$ and integrating between limits $x = 0$ to $x = a$; we get

$$\sum_n A_{mn} \sin \frac{n\pi y}{a} = \frac{2}{a} \int_0^a u_0 \sin \frac{m\pi x}{a}\, dx \quad \text{... (26)}$$

Again multiplying this equation with $\sin \dfrac{n\pi y}{b}$ and integrating between limits $y = 0$ to $y = b$; we obtain

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b u_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \qquad \dots (27)$$

Similarly, we have

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_0^a \int_0^b \dot{u}_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \, dx \, dy \qquad \dots (28)$$

With these values of A_{mn} and B_{mn}, equation (22) gives the required solution.

Ex. 22. *Find the displacement of a square membrane of unit length and unit wave velocity along it with initial velocity and initial amplitude as zero and A sin πx sin $2\pi y$ respectively.*

(Rohilkhand 2000)

Solution. The general solution of a vibrating rectangular membrane is given by

$$u = \sum_m \sum_n (A_{mn} \cos \omega_{mnt} + B_{mn} \sin \omega_{mnt}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \dots (1)$$

where $\omega_{mn} = \pi v \sqrt{\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}}$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b \dot{u}_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

and $$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_0^a \int_0^b \dot{u}_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

Here $a = b = 1$, $v = 1$ and $\dot{u}_0$ (initial velocity) = zero

$\therefore \quad \omega_{mn} = \pi \sqrt{m^2 + n^2}$ and $B_{mn} = 0$

Also given $u_0 = A \sin \pi x \sin 2\pi y$

$$A_{mn} = 4A \int_0^1 \int_0^1 \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y \, dx \, dy.$$ Integrating we get

$$A_{12} = 4A \int_0^1 \int_0^1 \sin \pi x \sin 2\pi y \sin \pi x \sin 2\pi y \, dx \, dy$$

$$= 4A \int_0^1 \int_0^1 \sin^2 \pi x \, dx \sin^2 2\pi y \, dy$$

$$= A \left\{ \int_0^1 (1 - \cos 2\pi x) \, dx \int_0^1 (1 - \cos 4\pi y) \, dy \right\}$$

$$= A \times 1 \times 1 = A$$

All other coefficients A_{mn} are zero

$\therefore\ \omega_{mn} = \pi\sqrt{m^2 + n^2} = \pi\sqrt{(1)^2 + (2)^2} = \sqrt{5}\,\pi$. hence equation (1) takes the form

$$u = A\cos\sqrt{5}\,\pi t \sin \pi x \sin 2\pi y$$

This is required expression.

9·19. Normal Modes in Three Dimensions

Consider sound waves confined in a rectangular box of sides a, and c. The boundaries defined by coordinate axes $x = 0$, $y = 0$, $z = 0$ and the lines $x = a$, $y = b$ and $z = c$. The general equation of waves is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2\phi}{\partial x^2} \quad \text{...}$$

where u is the speed of waves. Clearly ϕ is the function of x, y, z and t. By the method of separation of variables let

$$\phi(x, y, z, t) = X(x)\,Y(y)\,Z(z)\,T(t) \quad \text{...}$$

where X, Y, Z and T are the functions of x, y, z and t only respectively.

Substituting (2) in (1) and dividing throughout by $X(x)\,Y(y)\,Z(z)\,T(t)$, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial t^2} = \frac{1}{u^2}\frac{\partial^2 T}{\partial t^2}$$

$$\Rightarrow \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{u^2 T}\frac{\partial^2 T}{\partial t^2} = -\frac{1}{X}\frac{\partial^2 X}{\partial x^2} \quad \text{...}$$

In this equation L. H. S. is the function of y, z, t but independent of x while R.H.S. is function of x only ; therefore each side must be equal to a constant say $(+k_1^2)$

$$\therefore \quad -\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = +k_1^2 \quad \text{...}$$

$$\text{and} \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{u^2 T\partial t^2}\partial^2 T = k_1^2 \quad \text{...}$$

Equation (5) may be expressed as

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{u^2 T}\frac{\partial^2 T}{\partial t^2} = -k_1^2 - \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad \text{...}$$

In this equation L. H. S. is the function of z, t and is independent of y and R. H. S. is a function of y ; so each side should be equal to a constant k_2^2 (say)

$$\Rightarrow \quad -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_2^2 \text{ and} \quad \text{...}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{1}{u^2 T}\frac{\partial^2 T}{\partial t^2} - k_1^2 = k_2^2 \quad \text{...}$$

Equation (8) may be expressed as

$$-\frac{1}{u^2 T} + \frac{\partial^2 T}{\partial t^2} - k_1^2 - k_2^2 = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \quad \text{...}$$

In this equation L. H. S. is the function of t only, while R. H. S is the function of z only so each side must be equal to a constant k_3^2 (say).

$$\Rightarrow \quad -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k_3^2 \quad \text{... (1}$$

$$\text{and} \quad -\frac{1}{u^2 T}\frac{\partial^2 T}{\partial t^2} - k_1^2 - k_2^2 = k_3^2 \quad \text{... (1}$$

Now equations (4), (7), (10) and (11) may be expressed as

$$\frac{\partial^2 X}{\partial x^2} + k_1^2 X = 0 \quad \text{... (12)}$$

$$\frac{\partial^2 Y}{\partial y^2} + k_2^2 Y = 0 \quad \text{... (13)}$$

$$\frac{\partial^2 Z}{\partial z^2} + k_3^2 Z = 0 \quad \text{... (14)}$$

and $$\frac{\partial^2 T}{\partial t^2} + u^2 k^2 T = 0 \quad \text{... (15)}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$

the solutions of equations (12), (13), (14) and (15) are

$$X = A_1 e^{\pm i k_1 x}$$

$$Y = A_2 e^{\pm i k_2 y}$$

$$Z = A_3 e^{\pm i k_3 z}$$

$$T = A_4 e^{\pm i u \kappa \tau}$$

Thus the final solution is

$$\phi = X\, Y\, Z\, T$$

or $$\phi = A\, e^{\pm i k_1 x}\, e^{\pm i k_2 y}\, e^{\pm i k_3 z}\, e^{\pm i u k t} \quad \text{... (16)}$$

where $A_1 A_2 A_3 A_4 = A$ (*a* new constant)

Equation (16) may be expressed as

$$\phi = A\,(\cos k_1 x \pm i \sin k_1 x)\,(\cos k_2 y \pm i \sin k_2 y)\,(\cos k_3 z \pm i \sin k_3 z)\, e^{\pm i u k t} \quad \text{... (17)}$$

Using boundary condition $\phi = 0$ along $x = 0$, $y = 0$, $z = 0$, we note that the coefficients of cosine terms involving x, y, z, will be zero and using $uk = \omega$ where ω is angular frequency, equation (17) takes the form

$$\phi = A \sin k_1 x \sin k_2 y \sin k_3 z\, e^{\pm i \omega t} \quad \text{... (18)}$$

Again $\phi = 0$ at $x = a$, $y = b$ and $z = c$ gives

$\sin k_1 a = 0$, $\sin k_2 b = 0$ and $\sin k_3 c = 0$

$\Rightarrow \quad k_1 a = n_1 \pi$, $k_2 b = n_2 \pi$ and $k_3 c = n_3 \pi$ where n_1, n_2, n_3 are integers

Thus we have

$$k_1 = \frac{n_1 \pi}{a},\ k_2 = \frac{n_2 \pi}{b} \text{ and } k_3 = \frac{n_3 \pi}{c} \quad \text{... (19)}$$

Thus $$k^2 = k_1^2 + k_2^2 + k_3^2$$

or $$k^2 = \pi^2 \left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right) \quad \text{... (20)}$$

As $$k = \frac{2\pi}{\lambda}$$

$$\Rightarrow \quad \left(\frac{2\pi}{\lambda}\right)^2 = \pi^2 \left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)$$

$$\Rightarrow \quad \frac{2}{\lambda} = \sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}} \quad \text{... (21)}$$

As $\lambda = \frac{u}{\nu}$, where ν is frequency, equation (21) gives

$$\nu = \frac{u}{2}\sqrt{\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}} \quad \text{... (22)}$$

This equation gives the normal modes of vibrations in three dimensions. Each combination of (n_1, n_2, n_3) is called a mode.

The eigen functions ϕ are given by

$$\phi = \sum_{n_1}\sum_{n_2}\sum_{n_3} A_{n_1, n_2, n_3} \sin\frac{n_1\pi x}{a} \sin\frac{n_2\pi y}{b} \sin\frac{n_3\pi z}{c} e^{\pm i\omega t} \quad \text{... (23)}$$

9·20. The Vibrations of Circular Membrane

The basic equation of vibration of a membrane in two dimensions is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2} \quad \text{... (1)}$$

In the case of circular membrane, it is convenient to use polar coordinates (r, θ) *viz.*,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Transforming equation (1) from cartesian to polar coordinates ; we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2} \quad \text{... (2)}$$

Now let us consider *symmetric case* where the motion is started in a symmetric manner about the origin so that the displacement u of membrane is a function of r and t and is independent of angle θ. Then the displacement

$$u = u(r, t) \quad \text{... (3)}$$

obviously, $\frac{\partial u}{\partial \theta} = 0$; hence equation (2) in the case of *symmetrical vibrations* takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2} \quad \text{... (4)}$$

By the method of separation of variables, let us substitute

$$u(r, t) = R(r)\,\tau(t) \quad \text{... (5)}$$

where R is a function of r only and τ is a function of t only. Substituting (5) in (4) and dividing by $R\tau$; we get

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \frac{1}{\nu^2\tau}\frac{\partial^2 \tau}{\partial t^2} \quad \text{... (6)}$$

In this equation left hand side is a function of r only and right hand side is a function of t only ; therefore each side must be equal to the same constant $-\frac{\omega^2}{\nu^2}$ say, *i.e.*,

$$\frac{1}{R}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{\omega^2}{\nu^2} \quad \text{... (7)}$$

and

$$\frac{1}{\nu^2\tau}\frac{\partial^2 \tau}{\partial t^2} = -\frac{\omega^2}{\nu^2} \quad \text{or} \quad \frac{\partial^2 \tau}{\partial t^2} + \omega^2\tau = 0. \quad \text{... (8)}$$

Equation (8) in time function has the same form as for a rectangular membrane. Hence its solution is given by

$$\tau = Ae^{\pm i\omega t} = A_1\cos\omega t + A_2\sin\omega t \quad \text{... (9)}$$

Equation (7) may be expressed as

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{\omega^2}{v^2} = 0$$

or
$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \frac{\omega^2}{v^2} R = 0$$

i.e.
$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R = 0 \quad \text{... (10)}$$

where
$$k = \frac{\omega}{v} \quad \text{... (11)}$$

Equation (8) is a *Bessel' equation* of zeroth order ; for which the general of solution is given by

$$R = B\, J_0\,(kr) + C\, Y_0\,(kr) \quad \text{... (12)}$$

where B and C are arbitrary constants ; J_0 and Y_0 are Bessel's functions of zeroth order of first and second kind respectively.

Since the amplitude of vibration is finite at the origin $r = 0$; therefore R must be finite at $r = 0$. But $Y_0\,(kr) \to \infty$ as $r \to 0$; hence we must have $C = 0$. Equation (12) then gives

$$R = B\, J_0\,(kr) \quad \text{... (13)}$$

If a is the radius of the membrane and the periphery of the membrane is fixed, then amplitude of vibration must be zero at $r = a$; *i.e.,*

$$B\, J_0\,(ka) = 0 \quad \text{or} \quad J_0\,(ka) = 0 \quad \text{... (14)}$$

(Since $B \neq 0$)

From the table of Bessel's functions, we note that equation (14) is satisfied if ka has values 2·404, 5·520, 8·653, 14·93 etc. Accordingly the *fundamental angular frequency* ω_1 is given by

$$\omega_1 = kv = \frac{2{\cdot}404v}{a} \quad \text{... (15)}$$

The other possible angular frequencies may be given by using other values of ka given above. It is obvious that there are infinite number of natural frequencies possible, which are not multiples of each other.

If possible solutions of ω are denoted by ω_n, then the general solution (11) for R may be expressed as

$$R = \sum_n B_n\, J_0\,(k_n r) = \sum_n B_n\, J_0\left(\frac{\omega_n r}{v}\right) \quad \text{... (16)}$$

∴ The complete solution is given by

$$u = R\tau = \sum_{n=1}^{\infty} (A_1 \cos \omega_n t + A_2 \sin \omega_n t)\, B_n\, J_0\left(\frac{\omega_n r}{v}\right)$$

$$= \sum_{n=1}^{\infty} (C_n \cos \omega_n t + D_n \sin \omega_n t)\, J_0\left(\frac{\omega_n r}{v}\right) \quad \text{... (17a)}$$

$$= \sum_{n=1}^{\infty} (C_n \cos v\, k_n t + D_n \sin v\, k_n t)\, J_0\,(k_n r) \quad \text{... (17b)}$$

$\left(\text{since } k_n = \frac{\omega_n}{v}\right)$

The arbitrary constants C_n and D_n may be evaluated by *initial conditions* that *initial displacement* for any value of r is $u = u_0(r)$ at $t = 0$ and *initial velocity* is $\frac{du}{dt} = \dot{u}, (r, 0) = \dot{u}_0$ at $t = 0$.

Substituting $t = 0$ and $u = u_0$ in (17b) ; we get

$$u_0 = \sum_n C_n J_0(k_n r) \quad \text{... (18)}$$

Multiplying both sides of this equation by $r\ J_0(k_m r)$ and integrating between limits 0 and a, we get

$$\int_0^a u_0\, r\, J_0(k_m r)\, dr = \sum_n C_n \int_0^a r J_0(k_m r) J_0(k_n r) dr = C_m \int_0^a r\{J_0(k_m r)\}^2 dr$$

(other terms vanishing)

$$= C_m \cdot \frac{a^2}{2} J_1^2(k_m a)$$

[using orthogonal property of Bessel's functions]

This gives

$$C_m = \frac{2}{a^2 J_1^2(k_m a)} \int_0^a u_0 \cdot r\, J_0(k_m r)\, dr$$

Setting $m = n$, we get

$$C_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a u_0 \cdot r\, J_0(k_n r)\, dr \quad \text{... (19)}$$

Now differentiating equation (17b) with respect to time ; we get

$$\dot{u} = \sum_{n=1}^{\infty} \{-C_n v k_n \sin v k_n t + v k_n D \cos v k_n t\} J_0(k_n r)$$

Now using $\dot{u} = \dot{u}_0$ at $t = 0$, we get

$$\dot{u}_0 = \sum v k_n D_n J_0(k_n r)$$

Multiplying again by $r J_0(k_m r)$ and integrating between limits 0 and a, we get

$$\int_0^a \dot{u}_0\, r\, J_0(k_m r)\, dr = \sum_n v k_n D_n \int_0^a r J_0(k_n r) J_o(k_m r)\, dr = v k_m D_m \int_0^a r J_0^2(k_m r)\, dr$$

$$= v k_m D_m \cdot \frac{a^2}{2} J_1^2(k_m a) \quad \text{(other integrals vanishing)}$$

This gives

$$D_m = \frac{2}{v k_m^2 a^2 J_1^2(k_m a)} \int_0^a \dot{u}_0\, r J_0(k_m r)\, dr$$

Setting $m = n$, we get

$$D_n = \frac{2}{v k_n a^2 J_1^2(k_n a)} \int_0^a \dot{u}_0\, r\, J_0(k_n r)\, dr \quad \text{... (20)}$$

With these values of C_n and D_n, equation (17a) gives required solution.