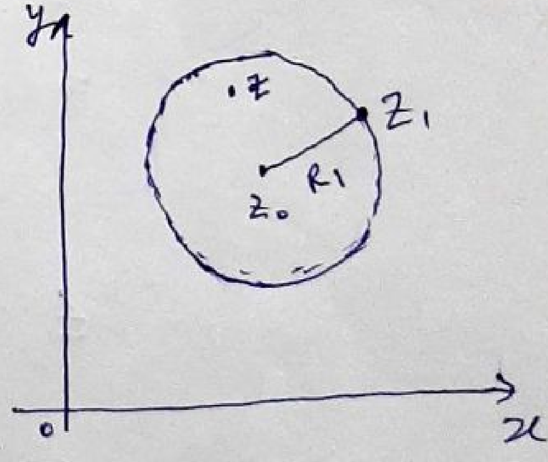
Section 63 :-

In the previous sections, we have seen the Taylor's and Laurent series representation of a function. In this section we discuss the absolute and uniform convergence of a power series.

Theorem 1 :- If a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad — (1)$$

converges when $z = z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point z in open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.



Proof :- As the power series (1) converges when $z = z_1$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n \text{ converges}$$

which is a series of complex numbers.

$\therefore$ its n^{th} term converges to 0 as n tend ∞.

$\Rightarrow$ $(a_n (z_1 - z_0)^n)$ is bounded,

$\therefore$ $\exists$ $M > 0$ such that

$$|a_n (z_1 - z_0)^n| \leq M \quad \forall\, n = 0, 1, 2, \ldots \quad — (2)$$

and if $|z-z_0| < R_1$

$\Rightarrow |z-z_0| < |z_1-z_0|$

$\Rightarrow \dfrac{|z-z_0|}{|z_1-z_0|} < 1$

Let $\rho = \dfrac{|z-z_0|}{|z_1-z_0|}$, then $\rho < 1$. — (3)

Consider,

$$|a_n(z-z_0)^n| = \left| a_n(z-z_0)^n \frac{(z_1-z_0)^n}{(z_1-z_0)^n} \right|$$

$$\cancel{= |a_n(z_1-z_0)^n \rho^n|}$$

$$= |a_n(z_1-z_0)^n| \frac{|(z-z_0)|^n}{|z_1-z_0|^n}$$

$$\leq M.\rho^n \qquad [\text{from eq. (2) \& (3)}].$$

$\therefore |a_n(z-z_0)^n| \leq M\rho^n$.

As $\sum M\rho^n$ is a geometric series, which converges since $\rho < 1$.

Therefore, By comparison test for series of real numbers.

$$\sum_{n=0}^{\infty} |a_n(z-z_0)^n| \text{ converges}.$$

whenever $|z-z_0| < R_1$.

This theorem tell us that if the power series (1) converges at some point z_1 other than z_0, then power series (1) is convergent for all points inside circle $|z-z_0| = R_1$ where $R_1 = |z-z_0|$.

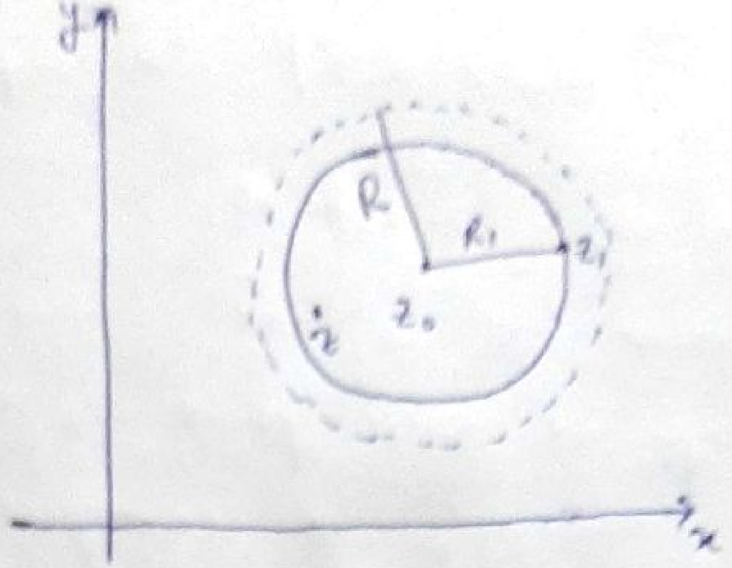
This implies that $|z-z_0| < R_1$ is the region of convergence for power series (1).

The greatest circle centered at z_0 such that (1) converges at each point inside is called the <u>Circle of convergence</u> of series (1). The series cannot converge at any point z_2 outside that circle.

<u>Theorem 2</u>:- If z_1 is a point inside the circle of convergence $|z-z_0| = R$ of a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad -(4)$$

then the series must be uniformly convergent in the closed disk $|z-z_0| \leq R_1$ where $R_1 = |z_1 - z_0|$.



Proof:- Let $S(z)$ and $S_N(z)$ represents the sum and partial sum, respectively of series ④.

$$\therefore \quad S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$S_N(z) = \sum_{n=0}^{N-1} a_n (z-z_0)^n \qquad . \; (|z-z_0|<R).$$

Then the remainder function of ④ is,

$$f_N(z) = S(z) - S_N(z)$$

$$= \sum_{n=N}^{\infty} a_n (z-z_0)^n.$$

To show that series ④ converges uniformly in $|z-z_0| \leq R_1$, we have to show that the sequence $(f_N(z))$ converges uniformly to 0 in $|z-z_0| \leq R_1$.

i.e. for $\epsilon > 0$, $\exists$ N_ϵ (depending only on ϵ, but not on z) s.t.

$$|f_N(z)| < \epsilon \quad \text{whenever } N > N_\epsilon$$

and for all z: $|z-z_0| \leq R_1$.

As z_1 is a point lying inside the circle of convergence of series (4), Thus By theorem (1),

$$\sum_{n=0}^{\infty} |a_n(z_1-z_0)^n| \quad \text{converges} \quad \text{--- (5)}.$$

Let σ_N be the remainder of series (5), then

$$\sigma_N = \sum_{n=N}^{\infty} |a_n(z_1-z_0)^n|$$

$$= \lim_{m\to\infty} \sum_{n=N}^{m} |a_n(z_1-z_0)^n|$$

Similarly, as $f_N(z) = \sum_{n=N}^{\infty} a_n(z-z_0)^n$

$$f_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z-z_0)^n$$

where m is positive integer such that $m > N$.

$$\Rightarrow \quad |f_N(z)| = \left| \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z-z_0)^n \right|$$

$$|f_N(z)| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n(z-z_0)^n \right| \quad \left[\because \text{If } \lim_{n\to\infty} z_n = z \text{ then } \lim_{n\to\infty} |z_n| = |z| \right]$$

and, when $|z-z_0| \le R_1 = |z_1-z_0|$

$$\left| \sum_{n=N}^{m} a_n(z-z_0)^n \right| \le \sum_{n=N}^{m} |a_n| \, |z-z_0|^n \quad [\text{By } \Delta^{\text{r}} \text{ inequality}]$$

$$\leq \sum_{n=N}^{m} |a_n| |z_1 - z_0|^n \qquad \left[\because |z-z_0| \leq R_1\right].$$

$$= \sum_{n=N}^{m} |a_n (z_1 - z_0)^n| .$$

$$\Rightarrow \left| \sum_{n=N}^{m} a_n (z-z_0)^n \right| \leq \sum_{n=N}^{m} |a_n (z_1 - z_0)^n|$$

$$\Rightarrow \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n (z-z_0)^n \right| \leq \lim_{m\to\infty} \sum_{n=N}^{m} |a_n (z_1 - z_0)^n|$$

$$\Rightarrow \quad |f_N(z)| \leq \sigma_N \quad \text{when } |z-z_0| \leq R_1 . \quad \text{— (6)}$$

As σ_N are remainders of a convergent series (5), they tend to 0 as $N \to \infty$.

$\Rightarrow$ for $\epsilon > 0$, $\exists$ $N_\epsilon \in \mathbb{N}$ s.t.

$$\sigma_N < \epsilon \quad \text{whenever } N > N_\epsilon .$$

$\therefore$ Using eq. (6), we have.

$$|f_N(z)| < \epsilon \quad , \text{whenever } N > N_\epsilon$$

and for all $z : |z - z_0| \leq R_1$

$\Rightarrow$ $(f_N(z))$ tends to 0 uniformly in $|z-z_0| \leq R_1$

$\Rightarrow$ series (4), converges uniformly in $|z-z_0| \leq R_1$.

■

Section: 66

We have done Taylor and Laurent series representation in previous sections, we will do the uniqueness of Taylor and Laurent series representation. We first consider the uniqueness of Taylor series representation.

Theorem 1:- If a series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad - (1)$$

converges to $f(z)$ at all points interior to some circle $|z-z_0|=R$, then it is the Taylor series expansion for f in the powers of $z-z_0$.

Proof:- To prove this theorem (i.e. uniqueness of Taylor's series), we will show that a_n in series (1) is nothing but Taylor's series coefficient.

i.e. $a_n = \frac{f^{(n)}(z_0)}{n!}$, $n=0,1,2,\ldots$

As series (1) converges to $f(z)$.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z: |z-z_0|<R \quad - (2)$$

Write eq. (2) using the index summation m.

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \quad , \; z : |z-z_0| < R. \quad — (3)$$

Now let $g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}}$ for any

n, s.t. $n = 0, 1, 2, \ldots$

and let C be a circle centered at z_0 and with radius less than R, then g is cts at C.

Using eq. (3), and applying theorem (1) of section 65 (Page 214), we get

$$\int_C g(z) f(z)\, dz = \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m\, dz. \quad — (4)$$

Consider,

$$\int_C g(z) f(z)\, dz = \int_C \frac{1}{2\pi i} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$= \frac{f^{(n)}(z_0)}{n!}$$

[Using extension of Cauchy Integral formula]

$$\therefore \int_C g(z) f(z)\, dz = \frac{f^{(n)}(z_0)}{n!} \quad — (5)$$

Consider,

$$\sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m dz = \sum_{m=0}^{\infty} a_m \int \frac{(z-z_0)^m}{(z-z_0)^{n+1}} dz$$

$$= \sum_{m=0}^{\infty} \frac{a_m}{2\pi i} \int_C \frac{dz}{(z-z_0)^{n-m+1}} \quad -(6)$$

Now consider,

$$\int_C \frac{dz}{(z-z_0)^{n-m+1}} = \begin{cases} 0, \text{ when } m \neq n \\ 2\pi i, \text{ when } m = n \end{cases}$$ [See Q10 of sec. 42]

$\therefore$ eq. (6) becomes,

$$\sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m dz = \frac{a_n}{2\pi i} \cdot 2\pi i = a_n \quad -(7)$$

$\therefore$ By eq. (5) and eq. (7), eq. (4) becomes

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

which is required.

Now we prove the uniqueness of Laurent's series.

Theorem 2 :- If a series

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad -(1)$$

converges to $f(z)$ at all points in some annular domain about z_0, then it is Laurent series expansion for f in powers of $(z-z_0)$ for that domain.

Proof:- ~~The~~ To prove the theorem we need to show that $$C_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad n=0,\pm1,\pm2,\ldots$$

As series (1) converges to $f(z)$

$$\therefore \quad f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad -(2)$$

$\underbrace{z : R_1 < |z-z_0| < R_2}_{\text{annular domain}}$.

Now let

$$g(z) = \frac{1}{2\pi i}\cdot\frac{1}{(z-z_0)^{n+1}} \qquad -(3)$$

for any integer n, s.t. $n=0,\pm1,\pm2,\ldots$

and let C be any circle inside the annulus, centered at z_0.

Then $g(z)$ is continuous on C.

Using eq. (2), apply theorem 1 (section 65) on page 214, we get

$$\cancel{\int_C g(z) f(z)\,dz = \sum_{m=-\infty}^{\infty} c_m (z-z_0)^m}$$

$$\int_C g(z) f(z)\,dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-z_0)^m\,dz \qquad -(4)$$

Consider,

$$\int_C g(z) f(z)\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad (\text{using } 3) \quad -5$$

and Consider

$$\int_C g(z)(z-z_0)^m dz = \frac{1}{2\pi i}\int_C \frac{dz}{(z-z_0)^{n-m+1}}$$

$$= \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

$$\therefore \sum_{m=-\infty}^{\infty} c_n \int_C g(z)(z-z_0)^m dz = c_n \quad -6$$

Using 5 and eq. 6, eq. 4 becomes.

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

which is required.