o.o $(\mathbb{Z}_4, \oplus_5)$ is group

(Q) $(M_{2\times 2}, +)$ is group where $\{M_{2\times 2}(R)\}$

(Q) $(M_{3\times 3}, \cdot)$ of 3×3 non-singular matrix
i.e $|A| \neq 0$ where $A \in M_{3\times 3}(R)$
is group under multiplication.

(Q) $(M_{2\times 2}, \cdot)$ is group wrt multiplication
where $|A| \neq 0$, $A \in M_{2\times 2}(R)$

$$M_{2\times 2}(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \begin{array}{l} a,b,c,d \in R \\ \text{\&} \quad ad - bc \neq 0 \\ \text{ie } |A| \neq 0 \end{array} \right\}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is group under multiplication.

(Q) $G = \{1, 2, 3, 4\}$ is group w.r.t multiplication modulo 5.

$(G, \otimes_5)$ where $a \otimes_5 b$ = remainder when $a \cdot b$ divided by 5.

(Sol^n)

$\otimes_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

All above entries are in G

$\therefore$ closure property holds

Associative law. Clearly $(a \otimes b) \otimes c = (a \otimes b) \otimes c$ $\forall a, b, c \in G$

Identity : $\exists\ 1 \in G$ s.t $a \otimes 1 = a \otimes 1 = a$ $\forall a \in G$

Inverse For each $a \in G$, $\exists\ a^{-1} \in G$ s.t

$a \otimes a^{-1} = a^{-1} \otimes a = 1$

Here, Inverse of 1 is 1

" " 2 is 3

" " 3 is 2

" " 4 is 4

$\therefore$ $(G, \otimes_5)$ is group w.r.t multiplication modulo 5.

(Q) $Z_n = \{0, 1, 2, \ldots, n-1\}$ for $n \geq 1$ is group under addition modulo n.

w.r.t addition modulo 6.

$Z_6 = \{0, 1, 2, 3, 4, 5\}$

closure property

$\oplus_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Note $a \oplus b$ = remainder when $a+b$ is divided by 6.

example $5 \oplus 3$ = remainder when 8 is divided by 6
$= 2$

All above entries are belongs to Z_6

$\therefore$ closure property holds

Associative law : we have
$(a \oplus b) \oplus c = a \oplus (b \oplus c) \ \forall \ a, b, c \in G$

Identity : $\exists \ 0 \in G$ s.t $a \oplus 0 = 0 \oplus a = a \ \forall \ a \in G$

Inverse : for each $a \in G$, $\exists \ a^{-1} \in G$ s.t
$a \oplus a^{-1} = a^{-1} \oplus a = 0$

Here inverse of 0 is 0
" " 1 " 5
" " 2 " 4
" " 3 " 3
" " 4 " 2
" " 5 " 1

$\therefore \{Z_6, \oplus_6\}$ is group

(9) Show that finite set $G = \{1, -1, i, -i\}$ is group wrt multiplication.

Soln) ① closure property

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

As all entries of above chart belongs to G

$\therefore$ closure property holds

② Associative law

we have

$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall \; a, b, c \in G$

③ Identity

$\exists \; 1 \in G$ s.t $1 \cdot a = a \cdot 1 = a \quad \forall \; a \in G$

④ Inverse, For each $a \in G$, $\exists \; \frac{1}{a} \in G$ s.t

$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$

since, inverse of 1 is 1

" " -1 is -1

" " i is $-i$

" " $-i$ is $-i$

Also, G is abelian group

since $a \cdot b = b \cdot a \quad \forall \; a, b \in G$

$\therefore (G, \cdot)$ is Abelian group.

(Exercise) $G = \{1, \omega, \omega^2\}$ w.r.t $\cdot$

where ω is complex cube root of unity

i.e $\omega^3 = 1$, $\omega^4 = \omega$, $\omega^5 = \omega^2$