Power Series:-

We have series of functions last time, now we will do a special kind of series of function 'Power series'.

Defn:- A series of real functions $\sum f_n$ is said to be a power series around $x=c$ if the function f_n has the form

$$f_n(x) = a_n (x-c)^n .$$

where a_n and c belong to $\mathbb{R}$ and where $n = 0, 1, 2, \ldots$

Note:- for the sake of simplicity of notation, we only treat the case where $c=0$.

Therefore By Power Series, we mean a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

Examples:-

(1) $\sum_{n=0}^{\infty} n!\, x^n$

This is a power series, here $a_n = n!$

(2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Here $a_n = \frac{1}{n!}$.

We wish to find an interval in which the power series $\sum a_n x^n$ converges. for that we first define the term called limit superior and limit inferior denoted as lim sup and lim inf respectively.

Defn:- Let (s_n) be a sequence in $\mathbb{R}$, we define limit superior as

$$\limsup s_n = \lim_{N \to \infty} \sup \{s_n : n > N\}.$$

and limit inferior as,

$$\liminf s_n = \lim_{N \to \infty} \inf \{s_n : n > N\}.$$

Note that in this definition we do not restrict (s_n) to be bounded. However we take $\sup\{s_n : n > N\} = +\infty$ for all N if (s_n) is not bounded above.

and take $\inf\{s_n : n > N\} = -\infty$ for all N if (s_n) is not bounded below.

Examples:-

(1.) Let $(a_n) = n$

i.e. $a_n = (1, 2, \ldots, n, \ldots)$

then $\limsup (a_n) = +\infty$

and $\liminf (a_n) = +\infty$.

(2.) Let $(a_n) = \left(5, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$

then $\limsup (a_n) = 0$

and $\liminf (a_n) = 0$

Note that $\sup (a_n) = 5$

and $\inf (a_n) = 0$

(3.) Let $(a_n) = 3 + \frac{1}{n}$

i.e. $a_n = \left(4, 3+\frac{1}{2}, 3+\frac{1}{3}, \ldots, 3+\frac{1}{n}, \ldots\right)$

then $\limsup (a_n) = 3$

$\liminf (a_n) = 3$

Note that $\sup (a_n) = 4$

and $\inf (a_n) = 3$.

Note:- (1) For any real sequence (a_n), $\limsup (a_n)$ may not equal to $\sup (a_n)$ [As seen in examples]

But $\limsup a_n \leq \sup (a_n)$.

(2) Also for any real valued sequence (s_n), lim inf(s_n) may not equal to inf(s_n).

But $\inf(s_n) \leq \liminf(s_n)$.

Exercise:-

Q.1. Find the lim sup and lim inf of the following.

(a) $(s_n) = (0,1,2,0,1,2,0,1,2, \text{---})$

(b) $s_n = 2 - \frac{1}{n}$

i.e $s_n = \left(2-\frac{1}{2}, 2-\frac{1}{3}, 2-\frac{1}{4}, \text{---} \cdots \right)$

(c) $s_n = (-1,-2,-3,\cdots\cdots)$.

Now we define the radius and interval of convergence for a Power Series $\sum a_n x^n$.

Def^n:- Let $\sum a_n x^n$ be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we define $\rho := \limsup(|a_n|^{1/n})$; if the sequence is not bounded we set $\rho = +\infty$. Then we define the radius of convergence of $\sum a_n x^n$ as

$$R := \begin{cases} 0 & \text{if } \rho = +\infty \\ 1/\rho & \text{if } 0 < \rho < \infty \\ +\infty & \text{if } \rho = 0 \end{cases}$$

then R is radius of convergence and the interval of convergence is $(-R, R)$ ..

<u>Thm</u> (<u>Cauchy - Hadamard Theorem</u>):- If R is the radius of convergence of the power series $\sum a_n x^n$, the series $\sum a_n x^n$ is absolutely convergent if $|x| < R$ and is divergent if $|x| > R$.

(Proof not in syllabus).

<u>Examples</u>:-

Determine the radius of convergence of the series $\sum a_n x^n$, where a_n is given by:-

① $1/n^n$

<u>Soln</u> Here the power series is $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$.

As
$$\begin{aligned} \rho &= \limsup \left(|a_n|^{1/n} \right) \\ &= \limsup \left(\left| \frac{1}{n^n} \right|^{1/n} \right) \\ &= \limsup \left(\frac{1}{n} \right) \\ &= 0 \end{aligned}$$

$\therefore$ Radius of convergence, $R = +\infty$.

(2) $a_n = 1$

Here power series is $\sum x^n$.

Then $\rho = \lim\sup((1)^{1/n})$

$= \lim\sup(1) = 1$

$\therefore \rho = 1$

$\Rightarrow$ Radius of convergence, $R = 1/\rho = 1$.

$\Rightarrow R = 1$.

(3.) $a_n = n^2$

$\therefore$ Power series is $\sum n^2 x^n$.

Here $\rho = \lim\sup(|n^2|^{1/n})$

$= \lim\sup(n^{2/n})$

$= 1$

$\Rightarrow \rho = 1$

$\Rightarrow$ Radius of convergence, $R = 1/\rho = 1$.

(4.) $a_n = n^n$

$\therefore$ Power series is $\sum n^n x^n$.

Here $\rho = \lim\sup(|n^n|^{1/n})$

$= \lim\sup(n) = +\infty$.

$\therefore \rho = +\infty$

$\Rightarrow$ Radius of convergence, $R = 0$.

Exercise:-

Determine the radius of convergence of the following power series.

(1.) $\sum n x^n$.

(2) $\sum n! x^n$

(3.) $\sum (n+1) x^n$

(4.) $\sum \frac{x^n}{n!}$.

Taylor Series:-

If a function f has ~~do~~ derivatives of all orders at a point $c \in \mathbb{R}$. Then f can be written as a power series around c as.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} \quad \text{for } |x-c| < R. \quad \text{--- (1)}$$

where $f^{(n)}$ denotes the n^{th} derivative.

In this case, we say that the power series (1) is the Taylor expansion of f at c.

Power series (Taylor expansion) of $\sin x$

Let $f(x) = \sin x$, $x \in \mathbb{R}$, then f has derivatives ~~derivatives~~ of all orders at 0.

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad -(1)$$

As $f(x) = \sin x$

$f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$

$f'(0) = 1$, $f''(x) = 0$

$f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$

$f^{(3)}(0) = -1$, $f^{(4)}(x) = 0$.

There fore, it we have

$$f^{(2n+1)}(x) = (-1)^n \cos x \quad \& \quad f^{(2n)}(x) = (-1)^n \sin x$$

$$\therefore \quad f^{(2n+1)}(0) = (-1)^n \quad \& \quad f^{(2n)}(0) = 0 .$$

Putting the values in eq. (1), we get

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} . x^{2n+1} .$$

$$\therefore \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad , \forall \, x \in \mathbb{R} .$$

Here the radius of convergence of the power series is $+\infty$.

As $\rho = \limsup \left(\left| \frac{(-1)^n}{(2n+1)!} \right|^{1/n} \right)$

$= \limsup \frac{1}{((2n+1)!)^{1/n}} = 0$

$\therefore \rho = 0$

$\Rightarrow$ Radius of convergence, $R = +\infty$.

Power Series (Taylor expansion) of $\cos x$:-

Let $f(x) = \cos x$, $x \in \mathbb{R}$

then $f^{(2n)}(x) = (-1)^n \cos x$ & $f^{(2n+1)}(x) = (-1)^n \sin x$

$\Rightarrow f^{(2n)}(0) = (-1)^n$ & $f^{(2n+1)}(0) = 0$.

$\therefore f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $x \in \mathbb{R}$.

Here also Radius of Convergence is $+\infty$.

Power Series (Taylor expansion) of e^x

Let $f(x) = e^x$, $x \in \mathbb{R}$, then f has derivatives of all order at 0.

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n.$$

As $f(x) = e^x$

$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$\vdots$$
$$f^{(n)}(x) = e^x$$

$$\therefore \quad f^{(n)}(0) = 1.$$

$$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}$$

Here $a_n = \frac{1}{n!}$

and $\rho = \limsup \left(\frac{1}{n!}\right)^{1/n} = 0$

therefore, Radius of convergence, $R = +\infty$.