Section 2-2

In the previous section we have studied linear transformations. In this section we will do ~~and~~ the representation of linear transformation by a matrix. In fact, there is a one-to-one correspondance between matrices and linear transformations. first we see some definitations.

Def^n:- Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.

Example:-

(1) Consider $\mathbb{F}^3$

then $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$ can be ~~xx~~ considered as ordered basis.

Also, $\gamma = \{(0,1,0), (1,0,0), (0,0,1)\}$ is also an ordered basis

But $\beta \neq \gamma$ as ordered basis.

(2.) for vector space F^n, we call $\{e_1, e_2, \dots, e_n\}$ the standard ordered basis for F^n.

(3.) for the vector space $P_n(F)$, we call $\{1, x, x^2, \dots, x^n\}$ the standard ordered basis for $P_n(F)$.

Defn:- Let $\beta = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$, let $a_1, a_2, \dots, a_n$ be unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i .$$

Then we define the coordinate vector of x relative to β, denoted by $[x]_\beta$, as

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Examples:-

(1.) Let $V = \mathbb{R}^2$ and $\beta = \{e_1, e_2\}$ be standard ordered basis.
i.e. $\beta = \{(1,0), (0,1)\}$

and let $x = (2,3) \in \mathbb{R}^2$, then

$$(2,3) = 2e_1 + 3e_2$$

$$\therefore \quad [(2,3)]_\beta = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(2) Let $V = P_2(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ be the standard ordered basis for V.

Let $f(x) = 4 + 6x - 7x^2$, then

$$f(x) = 4.1 + 6.x + (-7).x^2$$

$$\therefore \quad [f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Def^n:- Suppose that V and W are finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. Let $T: V \to W$ be linear. Then for each j, $1 \le j \le n$, there exists unique scalars $a_{ij} \in F$, $1 \le i \le m$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } 1 \le j \le n.$$

We call $m \times n$ matrix A defined by $A_{ij} = a_{ij}$. the the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

Note:- If $U: V \to W$ be a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$, then $U = T$ (why?)

Examples:-

(1.) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$$

and let β and γ be standard ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Compute $[T]_\beta^\gamma$.

→ As $T(e_1) = T(1,0) = (2, 3, 1)$

$$= 2.e_1 + 3.e_2 + 1.e_3.$$

and $T(e_2) = T(0,1) = (-1, 4, 0)$

$$= -1.e_1 + 4.e_2 + 0.e_3.$$

Thus $[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$.

(2.) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$ and $\beta = \{e_1, e_2\}$ and $\gamma = \{e_3, e_2, e_1\}$ are ordered basis for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, Compute $[T]_\beta^\gamma$.

$\rightarrow \quad T(e_1) = T(1,0) = (1, 0, 2)$
$= 2.e_3 + 0.e_2 + 1.e_1$

and $T(e_2) = T(0,1) = (3, 0, -4)$
$= -4.e_3 + 0.e_2 + 3e_1$.

$$\therefore \quad [T]_\beta^\gamma = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$$

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(3) Define $T: M_{2\times 2}(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$$

let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and

$\gamma = \{1, x, x^2\}$ are ordered bases, compute $[T]_\beta^\gamma$.

Sol^n: $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1+0) + (2.0)x + 0.x^2$

$= 1.1 + 0.x + 0.x^2$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0.x + 1.x^2$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.1 + 0.x + 0.x^2$

and $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0.1 + 1.x + 0.x^2$

$\therefore \quad [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Q5(b)

(4.) Define $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2.f(1) \\ 0 & f''(3) \end{pmatrix}$$

and $\beta = \{1, x, x^2\}$ and

$$\gamma = \left\{ \underset{E_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{E_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{E_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{E_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

are ordered bases. Compute $[T]_\beta^\gamma$.

Sol^n

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0.E_1 + 2.E_2 + 0.E_3 + 0.E_4.$$

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1.E_1 + 2.E_2 + 0.E_3 + 0.E_4$$

$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0.E_1 + 2.E_2 + 0.E_3 + 2.E_4$$

$$\therefore \quad [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Defn:- Let $T, U : V \to W$ be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define $T+U : V \to W$ by

$(T+U)(x) = T(x) + U(x)$ for all $x \in V$. (*)

and $aT : V \to W$ by $(aT)(x) = aT(x)$ $\forall x \in V$.

Theorem 2.7:- Using the operations of addition and scalar multiplication defined in above definitation, the collection of all linear transfor-mations from V to W is a vector space over F and is denoted by $\mathcal{L}(V, W)$. (In case $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, V)$).

Proof:- Let $\mathcal{L}(V, W) = \{T : V \to W : T \text{ is linear}\}$.

We want to show that $\mathcal{L}(V, W)$ is a vector space over F with operations defined on (*).

We first show that the operations defined here are closed under addition & scalar multiplication for that we want to show that

$$aT + U \in \mathcal{L}(V, W) \text{ if } T, U \in \mathcal{L}(V, W) \text{ and } a \in F.$$

Let $x, y \in V$ and $c \in F$, then

$$(aT+U)(cx+y) = aT(cx+y) + U(cx+y)$$

$= a[T(cx+y)] + cU(x) + U(y)$

$= a[cT(x) + T(y)] + cU(x) + U(y)$

$= acT(x) + aT(y) + cU(x) + U(y)$

$= c[aT(x) + U(x)] + aT(y) + U(y)$

$= c(aT+U)(x) + (aT+U)(y)$

$\therefore \quad (aT+U)(cx+y) = c(aT+U)(x) + (aT+U)(y).$

$\Rightarrow \quad aT+U$ is linear.

$\Rightarrow \quad aT+U \in \mathcal{L}(V,W)$.

$\therefore$ addition and scalar multiplication are closed in $\mathcal{L}(V,W)$.

Now we show that $\mathcal{L}(V,W)$ is a Vector space.

(VS1) let $T, U \in \mathcal{L}(V,W)$

then $(T+U)(x) = T(x) + U(x)$

$= U(x) + T(x) = (U+T)(x), \ \forall x \in V.$

$\Rightarrow \quad T+U = U+T$

(VS2) let $T, U, S \in \mathcal{L}(V,W)$, then

$((T+U)+S)(x) = (T+U)(x) + S(x)$

$= T(x) + U(x) + S(x)$

$= T(x) + (U+S)(x)$

$= (T+(U+S))(x), \ \forall x \in V.$

$\therefore (T+U)+S = T+(U+S).$

(VS3):- Define $T_0 : V \to W$ as
$$T_0(x) = 0$$
then T_0 is additive identity of $\mathcal{L}(V,W)$

(VS4) for $T \in \mathcal{L}(V,W)$, define $-T : V \to W$ as
$$(-T)(x) = -T(x).$$
then $-T \in \mathcal{L}(V,W)$ and is additive inverse of T.

(VS5) Let $T \in \mathcal{L}(V,W)$, then $(1.T)(x) = 1.T(x) = T(x)$
$$\Rightarrow \quad 1.T = T.$$

(VS6) Let $a, b \in F$ and $T \in \mathcal{L}(V,W)$, then
$$\begin{aligned}((ab)T)x = (ab)T(x) &= abT(x) \\ &= a(bT(x)) \\ &= (a(bT))(x) \quad \forall x \in V.\end{aligned}$$
$$\therefore \quad (ab)T = a(bT).$$

(VS7) Let $a \in F$ and $T, U \in \mathcal{L}(V,W)$
$$\begin{aligned}(a(T+U))(x) &= a(T+U)(x) \\ &= aT(x) + aU(x) \\ &= (aT + aU)(x) \quad \forall x \in V\end{aligned}$$
$$\therefore \quad a(T+U) = aT + aU.$$

(VS8) Let $a, b \in F$ and $T \in \mathcal{L}(V, W)$

then $((a+b)T)x = (a+b)T(x)$

$= aT(x) + bT(x)$

$= (aT + bT)(x) \quad \forall x \in V.$

$\therefore (a+b)T = aT + bT.$ ■

Thm 2.8 :- Let V and W be finite-dimensional vector spaces with order bases β and γ respectively, and let $T, U : V \to W$ be linear transformations. Then

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a.

Proof:- (a) Let $\beta = \{v_1, v_2, \ldots, v_n\}$

and $\gamma = \{w_1, w_2, \ldots, w_m\}$

then for every v_j $1 \le j \le n$, there exists unique scalars a_{ij} and b_{ij}, $1 \le i \le m$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

, for $1 \le j \le n$.

and $$U(v_j) = \sum_{i=1}^{m} b_{ij} w_i$$

then

$$(T+U)(v_j) = T(v_j) + U(v_j)$$
$$= \sum_{i=1}^{m} a_{ij} w_i + \sum_{i=1}^{m} b_{ij} w_i$$

$$(T+U)(v_j) = \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_i$$

Thus $$\left([T+U]_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = \left([T]_\beta^\gamma + [U]_\beta^\gamma\right)_{ij}$$

$$1 \le i \le n, \; 1 \le j \le m$$

$$\Rightarrow \quad [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma .$$

(b) As $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \qquad 1 \le j \le n$

then
$$(aT)(v_j) = aT(v_j)$$
$$= a \sum_{i=1}^{m} a_{ij} w_i$$
$$= \sum_{i=1}^{m} a a_{ij} w_i$$

$$\therefore \left([aT]_\beta^\gamma\right)_{ij} = a a_{ij} = a\left([T]_\beta^\gamma\right)_{ij}$$

$$\Rightarrow \quad [aT]_\beta^\gamma = a[T]_\beta^\gamma .$$

Example:-

(1) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 + 3a_2, \; 0, \; 2a_1 - 4a_2)$$

and $$U(a_1, a_2) = (a_1 - a_2, \; 2a_1, \; 3a_1 + 2a_2).$$

and β and γ are standard ordered bases of R^2 and R^3 respectively, then

$$\cancel{[T]^{\gamma}_{\beta} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix}} \qquad [T]^{\gamma}_{\beta} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

and $$[U]^{\gamma}_{\beta} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

and $(T+U)(a_1+a_2) = (2a_1+2a_2, 2a_1, 5a_1-2a_2)$

then $$[T+U]^{\gamma}_{\beta} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix}.$$

Verify that

$$[T+U]^{\gamma}_{\beta} = [T]^{\gamma}_{\beta} + [U]^{\gamma}_{\beta}.$$