$t \to \infty$ $t \to \infty$

C. Mixture Problems

We now consider rate problems involving mixtures. A substance S is allowed to flow into a certain mixture in a container at a certain rate, and the mixture is kept uniform by stirring. Further, in one such situation, this uniform mixture simultaneously flows out of the container at another (generally different) rate; in another situation this may not be the case. In either case we seek to determine the quantity of the substance S present in the mixture at time t.

Letting x denote the amount of S present at time t, the derivative dx/dt denotes the rate of change of x with respect to t. If IN denotes the rate at which S enters the mixture and OUT the rate at which it leaves, we have at once the basic equation

$$\frac{dx}{dt} = \text{IN} - \text{OUT} \tag{3.55}$$

from which to determine the amount x of S at time t. We now consider examples.

▶ **Example 3.10**

A tank initially contains 50 gal of pure water. Starting at time $t = 0$ a brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

1. How much salt is in the tank at any time $t > 0$?
2. How much salt is present at the end of 25 min?
3. How much salt is present after a long time?

Mathematical Formulation. Let x denote the amount of salt in the tank at time t. We apply the basic equation (3.55),

$$\frac{dx}{dt} = \text{IN} - \text{OUT}.$$

The brine flows in at the rate of 3 gal/min, and each gallon contains 2 lb of salt. Thus

$$\text{IN} = (2 \text{ lb/gal})(3 \text{ gal/min}) = 6 \text{ lb/min}.$$

Since the rate of outflow equals the rate of inflow, the tank contains 50 gal of the mixture at any time t. This 50 gal contains x lb of salt at time t, and so the concentration of salt at time t is $\frac{1}{50}x$ lb/gal. Thus, since the mixture flows out at the rate of 3 gal/min, we have

$$\text{OUT} = \left(\frac{x}{50} \text{ lb/gal}\right)(3 \text{ gal/min}) = \frac{3x}{50} \text{ lb/min}.$$

Thus the differential equation for x as a function of t is

$$\frac{dx}{dt} = 6 - \frac{3x}{50}. \tag{3.56}$$

Since initially there was no salt in the tank, we also have the initial condition

$$x(0) = 0. \tag{3.57}$$

Solution. Equation (3.56) is both linear and separable. Separating variables, we have

$$\frac{dx}{100 - x} = \frac{3}{50}\, dt.$$

Integrating and simplifying, we obtain

$$x = 100 + ce^{-3t/50}.$$

Applying the condition (3.57), $x = 0$ at $t = 0$, we find that $c = -100$. Thus we have

$$x = 100(1 - e^{-3t/50}). \tag{3.58}$$

This is the answer to question 1. As for question 2, at the end of 25 min, $t = 25$, and Equation (3.58) gives

$$x(25) = 100(1 - e^{-1.5}) \approx 78 \text{(lb)}.$$

Question 3 essentially asks us how much salt is present as $t \to \infty$. To answer this we let $t \to \infty$ in Equation (3.58) and observe that $x \to 100$.

▶ Example 3.11

A large tank initially contains 50 gal of brine in which there is dissolved 10 lb of salt. Brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the slower rate of 3 gal/min. How much salt is in the tank at any time $t > 0$?

Mathematical Formulation. Let $x =$ the amount of salt at time t. Again we shall use Equation (3.55):

$$\frac{dx}{dt} = \text{IN} - \text{OUT}.$$

Proceeding as in Example 3.10,

$$\text{IN} = (2 \text{ lb/gal})(5 \text{ gal/min}) = 10 \text{ lb/min};$$

also, once again

$$\text{OUT} = (C \text{ lb/gal})(3 \text{ gal/min}),$$

where C lb/gal denotes the concentration. But here, since the rate of outflow is different from that of inflow, the concentration is not quite so simple. At time $t = 0$, the tank contains 50 gal of brine. Since brine flows in at the rate of 5 gal/min but flows out at the slower rate of 3 gal/min, there is a net gain of $5 - 3 = 2$ gal/min of brine in the tank. Thus at the end of t minutes the amount of brine in the tank is

$$50 + 2t \text{ gal}.$$

Hence the concentration at time t minutes is

$$\frac{x}{50 + 2t} \text{ lb/gal},$$

and so

$$\text{OUT} = \frac{3x}{50 + 2t} \text{ lb/min}.$$

Thus the differential equation becomes

$$\frac{dx}{dt} = 10 - \frac{3x}{50 + 2t}. \tag{3.59}$$

Since there was initially 10 lb of salt in the tank, we have the initial condition

$$x(0) = 10. \tag{3.60}$$

Solution. The differential equation (3.59) is *not* separable but it *is* linear. Putting it in standard form,

$$\frac{dx}{dt} + \frac{3}{2t + 50} x = 10,$$

Scanned with CamScanner

we find the integrating factor

$$\exp\left(\int \frac{3}{2t+50}\,dt\right) = (2t+50)^{3/2}.$$

Multiplying through by this, we have

$$(2t+50)^{3/2}\frac{dx}{dt} + 3(2t+50)^{1/2}x = 10(2t+50)^{3/2}$$

or

$$\frac{d}{dt}[(2t+50)^{3/2}x] = 10(2t+50)^{3/2}.$$

Thus

$$(2t+50)^{3/2}x = 2(2t+50)^{5/2} + c$$

or

$$x = 4(t+25) + \frac{c}{(2t+50)^{3/2}}.$$

Applying condition (3.60), $x = 10$ at $t = 0$, we find

$$10 = 100 + \frac{c}{(50)^{3/2}}$$

or

$$c = -(90)(50)^{3/2} = -22{,}500\sqrt{2}.$$

Thus the amount of salt at any time $t > 0$ is given by

$$x = 4t + 100 - \frac{22{,}500\sqrt{2}}{(2t+50)^{3/2}}.$$

15. A tank initially contains 100 gal of brine in which there is dissolved 20 lb of salt. Starting at time $t = 0$, brine containing 3 lb of dissolved salt per gallon flows into the tank at the rate of 4 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

(a) How much salt is in the tank at the end of 10 min?

(b) When is there 160 lb of salt in the tank?

16. A large tank initially contains 100 gal of brine in which 10 lb of salt is dissolved. Starting at $t = 0$, pure water flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out at the slower rate of 2 gal/min.

(a) How much salt is in the tank at the end of 15 min and what is the concentration at that time?

(b) If the capacity of the tank is 250 gal, what is the concentration at the instant the tank overflows?

17. A tank initially contains 100 gal of pure water. Starting at $t = 0$, a brine containing 4 lb of salt per gallon flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture flows out at the slower rate of 3 gal/min.

(a) How much salt is in the tank at the end of 20 min?

(b) When is there 50 lb of salt in the tank?

18. A large tank initially contains 200 gal of brine in which 15 lb of salt is dissolved. Starting at $t = 0$, brine containing 4 lb of salt per gallon flows into the tank at the rate of 3.5 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture leaves the tank at the rate of 4 gal/min.

(a) How much salt is in the tank at the end of one hour?

(b) How much salt is in the tank when the tank contains only 50 gal of brine?

19. A 500 liter tank initially contains 300 liters of fluid in which there is dissolved 50 gm of a certain chemical. Fluid containing 30 gm per liter of the dissolved chemical flows into the tank at the rate of 4 liters/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the rate of 2.5 liters/min. How much of the chemical is in the tank at the instant it overflows?

20. A 200 liter tank is initially full of fluid in which there is dissolved 40 gm of a certain chemical. Fluid containing 50 gm per liter of this chemical flows into the tank at the rate of 5 liters/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the rate of 7 liters/min. How much of the chemical is in the tank when it is only half full?