Groups acting on themselves by left multiplication

Let G be any group and consider G acting on itself $(A = G)$ by left multiplication:

$$g \cdot a = ga \quad \forall \; g \in G, \; a \in G.$$

(If G has additive operation, then $g \cdot a = g + a$).

then for each fixed $g \in G$, the mapping

$\sigma_g : G \to G$ defined as

$\sigma_g(a) = ga$ is a permutation

and the map

$\phi : G \to S_G$ defined as

$\phi(g) = \sigma_g$ is homomorphism.

Now, let G is of finite order n,

~~furand~~ let $G = \{g_1, g_2, \ldots, g_n\}$

then we have,

$\sigma_g : G \to G$ as $\iff$ $\sigma_g : S_n \to S_n$ as

$\sigma_g(g_i) = gg_i = g_j$ $\qquad$ $\sigma_g(i) = j$

So, ~~we get a~~ this gives the permutation representation $G \to S_n$.

$$g \to \sigma_g$$

Ex:- Let $G = \{1, a, b, c\}$ Klein 4-group

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Label 1, a, b, c as 1, 2, 3, 4 respectively.

then we compute the permutation σ_a induced by a.

$a \cdot 1 = a1 = a$ and so $\sigma_a(1) = 2$

$a \cdot a = aa = 1$ and so $\sigma_a(2) = 1$

$a \cdot b = ab = c$ and so $\sigma_a(3) = 4$

$a \cdot c = ac = b$ and so $\sigma_a(4) = 3$

then $\sigma_a = (1\,2)(3\,4)$

$\therefore$ by homomorphism $\phi : G \to S_n$

$a \longrightarrow \sigma_a = (1\,2)(3\,4)$

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$b \cdot 1 = b1 = b$ and so $\sigma_b(1) = 3$

$b \cdot a = ba = c$ and so $\sigma_b(2) = 4$

$b \cdot b = bb = 1$ and so $\sigma_b(3) = 1$

$b \cdot c = bc = a$ and so $\sigma_b(4) = 2$

$\therefore$ $b \longrightarrow \sigma_b = (1\,3)(2\,4)$

$\therefore$ permutation representation $G \to S_4$ associated to this action is

$a \mapsto (12)(34)$ $\quad$ $b \mapsto (13)(24)$

$c \mapsto (14)(23)$.

Thm:- Let G be a gp acting on itself by left multiplication, then it always transitive and faithful and Stabilizer of any element is identity.

Proof:- Just prove faithful & Stab.

Now, let H be any subgp of G and let A be the set of left cosets of H in G.

then Define an action of G on A by.

$$g \cdot aH = gaH \quad \forall g \in G, aH \in A.$$

where gaH is the left coset with representative ga.

(i) $g_1 \cdot (g_2 \cdot aH) = g_1 \cdot (g_2 aH) = g_1 g_2 aH$

$(g_1 g_2) \cdot aH = g_1 g_2 aH$

(ii) $e \cdot aH = eaH = aH$

$\therefore$ G acts on the set of left cosets of H by left multiplication.

for $g \in G$, we have

$\sigma_g : A \to A$ as $\sigma_g(aH) = gaH$. permutati

$\phi : G \to S_A$ as $\phi(g) = \sigma_g$.

Now if $|H| = 1$, then the coset $aH = \{a\}$

then the action by left multiplication on left cosets of identity gp is same as the action of G on itself by left multiplication.

And if H is of finite index m in G.

i.e. A - set of left cosets of H in G is finite

$$A = \{ \underset{\substack{\downarrow \\ 1}}{a_1H}, \underset{\substack{\downarrow \\ 2}}{a_2H}, \ldots, \underset{\substack{\downarrow \\ m}}{a_mH} \}$$

Label them as $1 \quad 2 \quad \ldots \quad m$

In this way, we have a permutation $\sigma_g : S_m \to S_m$.

$\sigma_g(i) = j$ iff $ga_iH = a_jH$.

$\phi : g \to S_m$

$g \mapsto \sigma_g$.

__Ex:-__ $G = D_8$ $\quad H = \langle s \rangle$

then distinct left cosets are $1H, rH, r^2H, r^3H$.

Label them as $1, 2, 3, 4$ respectively.

$s.1H = sH = 1H$ and so $\sigma_s(1) = 1$

$s.rH = srH = r^3H$ and so $\sigma_s(2) = 4$

$s.r^2H = sr^2H = r^2H$ and so $\sigma_s(3) = 3$

$s.r^3H = sr^3H = rH$ and so $\sigma_s(4) = 2$

$s \downarrow \sigma_s = (2\ 4)$

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and $\sigma_r = (1\ 2\ 3\ 4)$

$r.1H = rH$ and so $\sigma_r(1) = 2$

$r.rH = r^2H$ $\sigma_r(2) = 3$

$r.r^2H = r^3H$ $\sigma_r(3) = 4$

$r.r^3H = r^4H = 1H$ $\sigma_r(4) = 1$

$$\sigma_r = (1\ 2\ 3\ 4)$$

$$r \longmapsto (1\ 2\ 3\ 4)$$

so, $\sigma_{sr} = \sigma_s \sigma_r$

$$= (2\ 4)(1\ 2\ 3\ 4)$$

$$= (1\ 2)(3\ 4)$$

<u>Thm</u>:- Let G be a gp. Let H be a subgp of G and let G act by left multiplication on the set A of left cosets of H in G. Let π_H be the associated permutation representation afforded by this action. Then

(i) G acts transitively on A.

(2) the stabilizer in G of the point $1H \in A$ is the subgp H.

(3) the kernel of the action (i.e. kernel of π_H) is $\bigcap_{x \in G} xHx^{-1}$, and $\ker \pi_H$ is the largest normal subgp of G contained in H.

Proof:- (i) Claim:- G acts transitively on A.

let aH and bH be two element of A.

We need to show that aH & bH are related

i.e. $\exists\, g \in G$ s.t. $g \cdot aH = bH$.

now let $g = ba^{-1}$

then $g \cdot aH = ba^{-1} \cdot aH = ba^{-1}aH = bH$

$\Rightarrow$ aH and bH lies in same orbit

$\Rightarrow$ G has only one orbit

$\Rightarrow$ G acts transitively on A.

(ii)

$$\begin{aligned} G_{1H} &= \{ g \in G \mid g \cdot 1H = 1H \} \\ &= \{ g \in G \mid gH = H \} \\ &= \{ g \in G \mid g \in H \} = H \end{aligned}$$

$\Rightarrow$ $G_{1H} = H$.

(iii)

$$\begin{aligned} \ker \pi_H &= \{ g \in G \mid gxH = xH \quad \forall\, x \in G \} \\ &= \{ g \in G \mid x^{-1}gxH = H \quad \forall\, x \in G \} \\ &= \{ g \in G \mid x^{-1}gx \in H \quad \forall\, x \in G \} \\ &= \{ g \in G \mid g \in xHx^{-1} \quad \forall\, x \in G \} \\ &= \bigcap_{x \in G} xHx^{-1}. \end{aligned}$$

and $\ker \pi_H \trianglelefteq G$ and $\ker \pi_H \leq H$; (check) [let $y \in \ker \pi_H$ $\Rightarrow y \in \bigcap_{x \in G} xHx^{-1}$ let $x = e$. $y \in H$.]

Let N is any normal subgp of G contained in H, then $N = xNx^{-1} \le xHx^{-1}$ $\forall$ $x \in G$

$\Rightarrow$ $N \le \bigcap_{x \in G} xHx^{-1} = \ker \pi_H$

$\Rightarrow$ $\ker \pi_H$ is the largest normal subgp of G contained in H.

Corollary :- (Cayley's theorem): Every group is isomorphic to a subgp of some symmetric gp. If G is a gp of order n, then G is isomorphic to a subgp of S_n.

Proof :- Let $|H| = 1$, then by above theorem the kernel of homomorphism G into S_G is contained in $H = 1$.

$\Rightarrow$ $|\ker \pi_H| = 1$ $\Rightarrow$ $\ker \pi_H = \{e\}$

$\Rightarrow$ $\phi : G \to S_G$ is one-one

and $\therefore$ $\phi : G \to \phi(G)$ is isomorphism

$\Rightarrow$ G is isomorphic to its image in S_G.

$\Rightarrow$ G is isomorphic to a subgp of symmetric gp.

Corollary :- If G is a finite gp of order n and p is the smallest prime dividing $|G|$, then any subgp of index p is normal.

Proof:- Given:- $|G| = n$, let H be a subgp of G s.t.
$p \mid |G|$ is the smallest prime, $|G:H| = p$

To Prove:- H is normal.

Let π_H be the associated permutation representation afforded by multiplication on the set of left cosets of H in G.

~~Then~~ Let $K = \ker \pi_H$
then $K \trianglelefteq H$ (By previous result)

Let $|H:K| = k$

then $|G:K| = |G:H||H:K| = pk$

As H has p left cosets, ($\Rightarrow |A| = p$ | A = set of left coset of H in G)

G/K is isomorphic to a subgp of S_p. (Cayley theorem)

$\Rightarrow$ $pk = |G/K|$ divides $p!$ [Langrange's theorem

$\Rightarrow$ k divides $(p-1)!$

As $p \nmid (p-1)!$ $\Rightarrow$ All prime divisors of $(p-1)!$ are less than p.

~~As~~ ~~p~~ and All prime divisors of k are ~~less~~ greater than p. (As p is smallest prime dividing $|G|$)

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Therefore $k = 1$

$\Rightarrow$ $|H : K| = 1$

$\Rightarrow$ $H = K = \ker \pi_H$

$\Rightarrow$ H is normal (As $\ker \pi_H$ is normal).

Groups Acting on themselves by Conjugation

Let G be a gp and it acts on itself by conjugation i.e.

$$g.a = gag^{-1} \quad \forall\, g \in G,\ a \in G.$$

Def^n:- Two elements a and b of G are said to be conjugate in G if there is some $g \in G$ such that $b = gag^{-1}$ (i.e. they are in the same orbit of G acting on itself by conjugation). The orbits of G acting on itself by conjugation are called the conjugacy class of G.

Ex:- (1) G is abelian gp
then $a \in G$, conjugacy class of a is $\{a\}$.

(2). If $|G|>1$, then G does not act transitively on itself by conjugation.

Now, the action by conjugation can be generalised

let a group G acts on the set $P(G) \to$ set of all subsets of G

s.t. $g \cdot S = gSg^{-1}$

where $gSg^{-1} = \{gsg^{-1} \mid s \in S\}$.

(i) $g_1 \cdot (g_2 \cdot S) = g_1 \cdot (g_2 S g_2^{-1})$

$= g_1 g_2 S (g_1 g_2)^{-1}$

(ii) $1 \cdot S = 1 S 1^{-1} = S$.

this defines a group action of G on $P(G)$.

<u>Def^n</u>:- Two subsets S and T of G are said to be conjugate in G if there is some $g \in G$ such that $T = gSg^{-1}$.

The no. of conjugates of S equals the index $|G : G_S|$ of the stabilizer G_S of S and

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S).$$

<u>Proposition 6.</u> The no. of conjugates of a subset S in a gp G is the index of the normalizer of S, $|G : N_G(S)|$. In particular, the no. of conjugate

of an element x of G is the index of centralizer of x, $|G : C_G(x)|$.

Theorem (The Class equation):
Let G be a finite gp and let $g_1, g_2, \ldots, g_r$ be representatives of the distinct conjugacy class of G not contained in the center $Z(G)$ of G. Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

Proof:- The element $\{x\}$ is a conjugacy class of size 1 iff $x \in Z(G)$ (done).

Now let $Z(G) = \{1, z_1, z_2, -, z_m\}$, and let $O_1, O_2, \ldots O_r$ be the conjugacy classes of G not contained in the center.

Now let g_i be a representative of O_i for each i.

Then all conjugacy classes are

$$\{1\}, \{z_1\}, \{z_2\}, \ldots \{z_m\}, O_1, O_2, \ldots, O_r$$

and
$$|G| = \sum_{i=1}^{m} 1 + \sum_{i=1}^{r} |O_i|$$
$$= |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|.$$

Thm 8. If p is prime and G is a group of prime power order p^{α} for some $\alpha \geq 1$, then G has a non-trivial center: $Z(G) \neq 1$.

Proof:- By Class equation,

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

where $g_1, \ldots, g_r$ are representatives of the distinct conjugacy classes of G not contained in $Z(G)$.

Now, if $C_G(g_i) = G$ for any i then done.

Therefore $C_G(g_i) \neq G$ for all $i = 1, 2, \ldots, r$.

$\Rightarrow$ $\underbrace{|C_G(g_i)|}_{= p^{\beta}} < |G|$ where $\beta < \alpha$.

$\therefore$ $|G : C_G(g_i)| = p^{\alpha - \beta}$

$\Rightarrow$ p divides $|G : C_G(g_i)|$ $\forall\, i = 1, 2, \ldots, r$

and also p divides $|G|$.

$\Rightarrow$ p divides $|Z(G)|$.

$\Rightarrow$ $Z(G)$ is non-trivial.

Corollary:- If $|G| = p^2$ for some prime p, then G is abelian. Moreover $G \cong Z_{p^2}$ or $\cong Z_p \oplus Z_p$

Proof:- By above theorem, $|Z(G)| \neq 1$.

$\Rightarrow |Z(G)| = p$ or p^2

then In both cases $\frac{G}{Z(G)}$ is cyclic $\left(\because \left|\frac{G}{Z(G)}\right| = p \text{ or } 1 \right)$

$$\Rightarrow \frac{G}{Z(G)} = \langle g Z(G) \rangle .$$

Now let $x \in G$ be arbitrary,

then $\quad xZ(G) = g^{\alpha} Z(G)$ for $\alpha \in \mathbb{Z}$.

$\Rightarrow (g^{\alpha})^{-1} x \cancel{x^{-1} g^{\alpha}} \in Z(G)$

$\Rightarrow (g^{\alpha})^{-1} x \cancel{x^{-1} g^{\alpha}} = z \quad$ for some $z \in Z(G)$

$\Rightarrow \quad x = g^{\alpha} z$

$\Rightarrow$ Every element of G can be written in form $g^{\alpha} z$

Now let $y \in G$, $y = g^{\beta} z_1$ where $z_1 \in Z(G)$

then $\quad xy = g^{\alpha+\beta} z z_1 = yx \quad$ { ~~there exist~~

$\Rightarrow \quad G$ is abelian.

And as $\quad |G| = p^2$

$$G \simeq \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_p \oplus \mathbb{Z}_p .$$

Result:- If $\frac{G}{Z(G)}$ is cyclic, then G is abelian.

Examples:-

① Consider $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

As $Z(D_8) = \{1, r^2\}$

and as $\langle g \rangle \leq C_G(g)$

$\therefore \langle r \rangle \leq C_G(r) \Rightarrow |C_G(r)| = 4$

$\Rightarrow |D_8 : C_G(r)| = 2 \Rightarrow r$ is related to r^3

$sr^2 = r^3 s (r^3)^{-1} = r^3 s r$ $\qquad r^3 = s r s^{-1} = s r s$

and $\{s, sr^2\}$, $\{sr, sr^3\} \rightarrow sr^3 = r^3 sr r$

then $|D_8| = 2 + 2 + 2 + 2 = 8$.

② Consider $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$.

$|Q_8| = 2 + 2 + 2 + 2 = 8$.

Conjugacy In S_n

Proposition 10:- Let σ, τ be elements of the symmetric gp S_n and suppose σ has cycle decomposition

$(a_1\, a_2 \ldots a_{k_1})(b_1\, b_2 \ldots b_{k_2}) \ldots$

then $\tau \sigma \tau^{-1}$ has cycle decomposition

$(\tau(a_1)\, \tau(a_2) \ldots \tau(a_{k_1}))(\tau(b_1)\, \tau(b_2) \ldots \tau(b_{k_2})) \ldots$