53 Liouville's theorem

Theorem 1 If a function f is entire and bounded in complex plane then $f(z)$ is constant througout the plane.

(proof) let f is entire and bounded function by using theorem 3 in Article 52

[(statement) suppose f is analytic inside and on a positively oriented circle C_R, centered at z_0 & radius R, then $|f^{(n)}(z_0)| \le \frac{n!\,M_R}{R^n}$ $(n=1,2,\dots)$ where M_R denotes maximum value of $|f(z)|$ on C]

Using it,

we have $|f'(z_0)| \le \frac{M_R}{R}$ (where $n=1$) — 1

Since, f is bounded

$\therefore$ there exists constant M s.t $|f(z)| \le M$ for all z.

Also, the constant M_R in inequality 1 is always less than or equal to M

we get $|f'(z_0)| \le \frac{M}{R}$ — 2 where R is arbitrarily large

Also, M is independent of value of R

letting $R \to \infty$ in eq^n 2, we get

$$|f'(z_0)| \le 0 \quad \forall\, z_0 \in \mathbb{C}$$

As $0 \le |f'(z_0)| \le 0 \quad \forall\, z_0 \in \mathbb{C}$

$\therefore f'(z_0) = 0 \quad \forall\, z_0 \in \mathbb{C}$

Since z_0 was arbitrary

$\therefore f'(z) = 0$ everywhere in Complex plane

Using theorem
If $f'(z) = 0$ everywhere in domain D
then $f(z)$ must be constant throughout D

$\therefore$ $f(z)$ is constant function. (Hence Proved)

Fundamental theorem of Algebra

Theorem (2) Any polynomial
$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$
of degree n $(n \geq 1)$ has atleast one zero.
i.e there exists atleast one point z_0 s.t $P(z_0) = 0$

(proof) Proof by contradiction
Suppose $P(z)$ is not zero for any value of z.
Then, the reciporal $f(z) = \frac{1}{P(z)}$

is clearly entire function

we have

$$|f(z)| = \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0|}$$

$$= \frac{1}{|z^n| \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|}$$

then $|f(z)| \to 0$ as $|z| \to \infty$

$\therefore$ for any arbitrary $\epsilon > 0$, $\exists$ $R > 0$ s.t

$|f(z)| < \epsilon$ for $|z| > R$ (however large) — (1)

Since $P(z)$ is polynomial fun^n

$\therefore$ $P(z)$ is cts function & $P(z) \neq 0$

$\therefore$ $f(z) = \frac{1}{P(z)}$ is continuous function

we get, $f(z)$ is continuous on every bounded region $|z| \le R$

$\Rightarrow$ $f(z)$ is bounded in $|z| \le R$

$\Rightarrow$ $\exists$ $k > 0$ s.t

$|f(z)| < k$ for $|z| \le R$ — (2)

from (1) & (2), we have

$|f(z)| \le M$ where $M = \max.\{\epsilon, k\}$ $\forall z$

$\Rightarrow$ f is bounded $\forall z$ & f is also entire function.

$\Rightarrow$ $f(z) = \frac{1}{P(z)}$ is constant function [By Liouville's theorem]

$\Rightarrow$ $P(z)$ is constant function

which is an contradiction

Hence $\exists$ $a \in \mathbb{C}$ s.t $P(a) = 0$ (Hence Proved)

[Remark Another proof of fundamental theorem of algebra is also given in Book] (Refer)(also)

(Remark) By Fundamental theorem of algebra, we have $P(z)$ has a zero say z_1,

then $P(z) = (z - z_1)\, Q_1(z)$ where $Q_1(z)$ is a polynomial of degree $n-1$, where $P(z)$ is poly. of degree n.

~~Algae~~ Again, by applying Fundamental theorem of algebra to $Q_1(z)$ polynomial, we have

$$P(z) = (z - z_1)(z - z_2)\, Q_2(z)$$

where $Q_2(z)$ is poly of degree $n-2$,

so on, we get

$$P(z) = (z - z_1)\,(z - z_2)\,(z - z_3) \cdots (z - z_n)\, C$$

where C is constant.

$\therefore$ $P(z)$ can have No more than n distinct zeros