Fundamental theorem of finite Abelian group

Theorem: Every finite Abelian group is direct product of cyclic groups of prime-power order. Moreover, the factorization is unique except for rearrangements of the factors.

In other words,
Every finite abelian group is direct product of cyclic groups of prime-power order. Moreover, the number of terms in the product & the orders of the cyclic groups are uniquely determined by the group.

The above theorems says, that

$$G \approx Z_{p_1^{n_1}} \oplus Z_{p_2^{n_2}} \oplus \cdots \oplus Z_{p_k^{n_k}} \quad -(1)$$

where p_i's are not necessarily distinct primes and the prime powers $p_1^{n_1}, p_2^{n_2} \cdots p_k^{n_k}$ are uniquely determined by G.
where G is finite abelian group

* Representation of group in above form (1) is called determining the isomorphism class of G.

(How it comes)
(1) Using every cyclic group of order n is $\approx Z_n$
So, cyclic group of prime-power order
say $p_i^{k_i} \approx Z_{p_i^{k_i}}$
In above theorem, then every cyclic group of prime power order is $\approx Z_{p_i^{k_i}}$

The Isomorphism classes of Abelian group of order p^k, where p is prime, $1 \le k \le 4$

Let $o(G) = p^k$, where p is prime
$1 \le k \le 4$
and G be finite abelian group.

There is one group of order p^k for each set of positive integers whose sum is k (such a set is called a partition of k) i.e

$k = n_1 + n_2 + \cdots + n_t$

where each n_i is a positive integer, then

$Z_{p^{n_1}} \oplus Z_{p^{n_2}} \oplus \cdots \oplus Z_{p^{n_t}}$

is an Abelian group of order p^k.

Order of G	Partitions of k	Possible direct products for G
p^1	1	Z_p
p^2	2	Z_{p^2}
	1+1	$Z_p \oplus Z_p$
p^3	3	Z_{p^3}
	2+1	$Z_{p^2} \oplus Z_p$
	1+1+1	$Z_p \oplus Z_p \oplus Z_p$
p^4	4	Z_{p^4}
	3+1	$Z_{p^3} \oplus Z_p$
	2+2	$Z_{p^2} \oplus Z_{p^2}$
	2+1+1	$Z_{p^2} \oplus Z_p \oplus Z_p$
	1+1+1+1	$Z_p \oplus Z_p \oplus Z_p \oplus Z_p$

Example

$p^4 = 5^4$

Order	partition of k	Possible direct products for G.
5^1	1	Z_5
5^2	2	Z_{5^2}
	1+1	$Z_5 \oplus Z_5$
5^3	3.	Z_{5^3}
	2+1	$Z_{5^2} \oplus Z_5$
	1+1+1	$Z_5 \oplus Z_5 \oplus Z_5$
5^4	4	Z_{5^4}
	3+1	$Z_{5^3} \oplus Z_{5^1}$
	2+2	$Z_{5^2} \oplus Z_{5^2}$
	2+1+1	$Z_{5^2} \oplus Z_5 \oplus Z_5$
	1+1+1+1	$Z_5 \oplus Z_5 \oplus Z_5 \oplus Z_5$

Remark:

The uniqueness portion of fundamental theorem guarantees that distinct partitions of k yield distinct isomorphism classes.

example: In above example

~~say order~~ order say 5^3

has distinct partition of k has distinct isomorphism classes

$Z_{5^2} \oplus Z_5 \not\approx Z_5 \oplus Z_5 \oplus Z_5$ (for partition of 3: 2+1, 1+1+1)

because $Z_{5^2} \not\approx Z_5 \oplus Z_5$

(using result ① If A is finite $A \oplus B \approx A \oplus C \Leftrightarrow B = C$)

(② Two Abelian groups are Isomorphic $\Leftrightarrow$ they have same number of elements of each order.)

The Isosmorphism classes of Abelian group of order n

Constructing all Abelian groups of certain order n where n has two or more distinct prime divisors.

i.e $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, n is prime - power decomposition.

Take $n = 1176$
$= 2^3 \cdot 3 \cdot 7^2$

Now consider 2^3, it is of form p^k, we have corresponding to 2^3, the isosmorphism classes

$$2^3 \longrightarrow \begin{array}{ll} 3 & Z_{2^3} \xrightarrow{\text{(written as)}} Z_8 \\ 2+1 & Z_{2^2} \oplus Z_{2^1} \xrightarrow{\text{(written as)}} Z_4 \oplus Z_2 \\ 1+1+1 & Z_2 \oplus Z_2 \oplus Z_2 \end{array}$$

Now,

$$3^1 \rightarrow 1 \qquad Z_3$$

$$7^2 \rightarrow \begin{array}{ll} 2 & Z_{7^2} \xrightarrow{\text{(written as)}} Z_{49} \\ 1+1 & Z_7 \oplus Z_7 \end{array}$$

∴ Complete list of distinct isosmorphism classes of Abelian groups of order 1176 is

$Z_8 \oplus Z_3 \oplus Z_{49}$

$Z_8 \oplus Z_3 \oplus Z_7 \oplus Z_7$

$Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_{49}$

$Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_7 \oplus Z_7$

$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_{49}$

$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_7 \oplus Z_7$