$i=1$

(using eq ①).

If we take $a_i = 1$ and $a_j = 0 \ \forall j \neq i$,

then $T(v_i) = w_i \quad \forall i = 1, 2, \ldots, n$ ——— ⑥

TS T is unique:

Suppose that $U : V \to W$ be linear such that $U(v_i) = w_i \quad \forall i = 1, 2, \ldots, n$

Then for $x \in V$ with linear combination of x as $x = \sum_{i=1}^{n} a_i v_i$,

$$U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i U(v_i)$$

$$= \sum_{i=1}^{n} a_i w_i = T(x)$$

$\therefore U(x) = T(x) \quad \forall x \in V$

$\Rightarrow U \equiv T$

$\exists$ such linear transformation T is unique.

Corollary:

Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \dots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \dots, n$, then $U = T$

Proof? Uniqueness part of above theorem.

Example: Define a L-T. $T : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$$

Suppose that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is linear such that $U(1, 0) = (-1, 3)$

and $U(1,1) = (1,3)$

then $U = T$.

Proof:- Clearly $\beta = \{(1,0), (1,1)\}$ is a basis for $\mathbb{R}^2$.

L-I-

$$a_1(1,0) + a_2(1,1) = 0$$

$$\Rightarrow (a_1 + a_2, a_2) = (0,0)$$

$$\Rightarrow \left.\begin{array}{l} a_1 + a_2 = 0 \\ a_2 = 0 \end{array}\right\} \Rightarrow a_1 = 0$$

β is L-I-

β generates $\mathbb{R}^2$.

Let $a_1(1,0) + a_2(1,1) = (a,b)$

$$\Rightarrow (a_1 + a_2, a_2) = (a,b)$$

$$\Rightarrow \left.\begin{array}{l} a_1 + a_2 = a \\ a_2 = b \end{array}\right\} \Rightarrow a_1 = a-b$$

$$\therefore (a,b) = (a-b)(1,0) + b(1,1)$$

β Spans $\mathbb{R}^2$.

Now, $T(1,0) = (2(0)-1, 3(1)) = (-1,3) = U(1,0)$

and $T(1,1) = (2(1)-1, 3(1)) = (1,3) = U(1,1)$

$\therefore T(1,0) = U(1,0)$ & $T(1,1) = U(1,1)$

$\Rightarrow T = U$ $\boxed{\text{or } T \equiv U}$