(6.4) DIMENSION THEOREM

Definition Kernel & Range

① Let V and W be two vector space and let $T: V \to W$ be L.T then the Kernel (called null space also) of T denoted by $Ker(T)$, is set of all $v \in V$ s.t $T(v) = O_W$ where O_W (zero vector of W)

$$Ker(T) = \{v \in V \text{ s.t } T(v) = O_W\}$$

② Range of T, denoted by $Range(T)$, is set of all vectors in W that are the image of some vectors in V

i.e $range(T) = \{w \in W \text{ s.t } w = T(v) \text{ for some } v \in V\}$

Theorem 6.9 Let $T: V \to W$ be L.T then $\ker(T)$ is a subspace of V and range(T) is subspace of W.

proof) $\ker(T) = \{v \in V;\ T(v) = 0_W\}$
$= T^{-1}\{0_W\}$

and Range $(T) = \{T(v);\ v \in V\}$
$= T(V)$

By theorem 6.3, we get $\ker(T)$ & range(T) are subspace of V & W respectively.

Theorem 6.10 Suppose that $T: V \to W$ is a L.T show that if $\{T(v_1), T(v_2), \ldots T(v_k)\}$ is L.I set of k distinct vectors in W, for some vectors $v_1, v_2, \ldots, v_k \in V$
then $\{v_1, v_2, \ldots v_k\}$ is L.I set in V.

(proof) Suppose that
$a_1 v_1 + \cdots + a_k v_k = 0_V$; $a_i \in \mathbb{R}$
$\Rightarrow T(a_1 v_1 + \cdots + a_k v_k) = T(0_V)$
$\Rightarrow a_1 T(v_1) + \cdots + a_k T(v_k) = 0_W$ ($\because T$ is L.T)
$\Rightarrow a_1 = a_2 = \cdots = a_k = 0$ ($\because T(v_1), T(v_2), \ldots, T(v_k)$ are L.I)

Remark: Converse of theorem 6.10

example: zero L.T
$T: \mathbb{R}^2 \to \mathbb{R}^2$ given by
$T([x_1, x_2]) = [0, 0]$ for all $[x_1, x_2] \in \mathbb{R}^2$

Now, $\{[1,0], [0,1]\}$ is L.I in $\mathbb{R}^2$
but $\{T[1,0], T[0,1]\} = \{[0,0]\}$
is Not L.I

Kernel Method

Method for finding Basis for Kernel of L.T

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be L.T given by $T(X) = AX$
where A is fixed $m \times n$ matrix
Kernel (T) consisting of all vectors X, that are solutions of $AX = 0$ (homogenous system)

step ① Find B, row reduced echelon form of A.

step ② Obtain set $\{v_1, v_2, \dots, v_n\}$ of particular solutions of Homogenous system $BX = 0$ by setting each independent variable in turn equal to 1 and all other independent variables to 0.

step ③ The set $\{v_1, v_2, \dots, v_n\}$ is basis for Ker(T).

Example ㉕ Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be L.T given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use Kernel Method to find basis for Ker(T).

(Solution) Step ① find B, row reduced echelon form of $A = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 & 11 \\ 2 & 1 & 8 \end{bmatrix}$$

the row reduced echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix}$$

Have column 3 as non-pivot

Hence, Homogenous system $BX = 0$ has only one Independent variable x_3
Setting $x_3 = 1$, gives particular solution
$v = [5, 2, 1]$
$\therefore$ $[-5, 2, 1]$ form basis for Ker(T).

Range Method :

Method for Finding a Basis for Range of a L.T

let $T : \mathbb{R}^n \to \mathbb{R}^m$ be L.T
given by $T(X) = AX$
where A is fixed $m \times n$ matrix & $X \in \mathbb{R}^n$

Step ① Find B, row reduced echelon form of A.

Step ② Form set of those columns of A whose corresponding columns in B have non-zero pivots.

The set formed is a basis for range(T).

Example (26)

Consider (L.T) $T : \mathbb{R}^5 \to \mathbb{R}^4$ given by

$T(X) = AX$ where

$$A = \begin{bmatrix} 8 & 4 & 16 & 32 & 0 \\ 4 & 2 & 10 & 22 & -4 \\ -2 & -1 & -5 & -11 & 7 \\ 6 & 3 & 15 & 33 & -7 \end{bmatrix}$$

Use Range method to find basis for range(T)

(Solution)

$$A = \begin{bmatrix} 8 & 4 & 16 & 32 & 0 \\ 4 & 2 & 10 & 22 & -4 \\ -2 & -1 & -5 & -11 & 7 \\ 6 & 3 & 15 & 33 & -7 \end{bmatrix}$$

Now, reduced row echelon form of A is

$$B = \begin{bmatrix} 1 & 1/2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Having non-zero pivots in columns 1, 3 & 5.
Hence corresponding columns of A form a basis for range(T)
i.e $\{[8, 4, -2, 6], [16, 10, -5, 15], [0, -4, 7, -7]\}$
forms a basis for range(T)

Remark: let V be finite dimensional V.S and $B = \{v_1, v_2, \ldots v_n\}$ be basis for V then Number of elements in B are called dimension of V.

Theorem (6.11) Dimension Theorem
If $T: V \to W$ be a L.T & V is finite dimension then range(T) is finite-dimensional and $\dim(\ker T) + \dim(\text{range}(T)) = \dim V$.

Example (27) Linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T([x_1, x_2, x_3]) = [0, x_2]$
find a basis for ker(T) & basis for range(T) and verify Dimension theorem.

(Solution) $\ker(T) = \{[x_1, x_2, x_3] ; T([x_1, x_2, x_3]) = [0,0]\}$

$= \{[x_1, x_2, x_3] ; [0, x_2] = [0,0]\}$

$= \{[x_1, x_2, x_3] ; x_2 = 0\}$

$= \{[x_1, 0, x_3] ; x_1, x_3 \in \mathbb{R}\}$

$= \{x_1 [1,0,0] + x_3 [0,0,1] ; x_1, x_3 \in \mathbb{R}\}$

thus span $\{[1,0,0], [0,0,1]\} = \ker(T)$

Hence, basis for $\ker(T) = \{[1,0,0], [0,0,1]\}$, since $\{[1,0,0], [0,0,1]\}$ is L.I.

$\therefore \dim(\ker(T)) = 2$

Also, range $(T) = \{[0, x_2] ; x_2 \in \mathbb{R}\}$

$= \{x_2 [0,1] ; x_2 \in \mathbb{R}\}$

the set of images obtained using T.

Thus range $(T) =$ span $\{[0,1]\}$

clearly $\{[0,1]\}$ is L.I

$\therefore$ Basis for range $(T) = \{[0,1]\}$

$\therefore \dim(\text{range}(T)) = 1$

Also, $\dim(\ker(T)) + \dim(\text{range}(T))$

$= 2 + 1 = 3 = \dim(\mathbb{R}^3)$.

Example (32) let $T : P_2 \to \mathbb{R}^2$ be L.T given by

$T(p) = [p(1), p'(1)]$

Find basis for $\ker(T)$ & basis for range(T).

Verify dimension theorem.

(14)

(Solution) P_2 (set of all polynomial p of degree ≤ 2)

$Ker(T) = \{p \in P_2$ s.t $[p(1), p'(1)] = [0, 0]\}$

$= \{p \in P_2$ s.t $p(1) = 0$ & $p'(1) = 0\}$

As $p(1) = 0$ ⟹ p must of form $(x-1)(ax+b)$ for some $a, b \in \mathbb{R}$

Differentiating p gives

$p' = (x-1)(a) + (ax+b)(1)$

Also $p'(1) = (1-1)(a) + a(1) + b$

$= a + b$

Also $p'(1) = 0$

⟹ $a + b = 0$

⟹ $a = -b$

∴ $p = (x-1)(ax+b)$ (∵ $a = -b$)

$= (x-1)(ax - a)$

$= a(x^2 - 2x + 1)$

In other words, $Ker(T) = \{a(x^2 - 2x + 1); a \in \mathbb{R}\}$

$= span(\{x^2 - 2x + 1\})$

∴ $dim(Ker(T)) = 1$

basis for $Ker(T)$ is $\{x^2 - 2x + 1\}$

A spanning set for range(T) can be find by image of a basis for P_2 using T.

Now, $T(x^2) = [(1)^2, 2(1)]$ using $T(p) = [p(1), p'(1)]$

$= [1, 2]$

$T(x) = [1, 1]$

$T(1) = [1, 0]$

Set $\{[1, 2], [1, 1]\}$ is L.I & $dim(\mathbb{R}^2) = 2$

Hence basis for range(T) = $\{[1, 2], [1, 1]\}$

$\dim(\text{range}(T)) = 2$

$\therefore \dim(\ker(T)) + \dim(\text{range}(T))$

$= 1 + 2 = 3 = \dim(P_2)$