$1, 3$ $2, 4, 5, 6, 7, \ldots, n$

$(n-2)!$

$n-2$ $(n-2)! \to S_n$

$A_n \to \dfrac{(n-2)!}{2}$

Chapter 10

Group Homomorphism:

A homomorphism ϕ from a group G to a group $\overline{G}$ is a function (or mapping) from G to $\overline{G}$ that preserves the group operation.

that is, $\phi(ab) = \phi(a)\phi(b) \ \forall a, b \in G$.

Example 1 $G = GL(2, \mathbb{R})$, $\overline{G} = \mathbb{R}^*$,

$\phi : G \to \overline{G}$ such that $\phi(A) = \det(A)$

$\phi(AB) = \det.(AB) = \det.(A)\det.(B) = \phi(A)\phi(B)$

$\therefore$ ϕ is a homomorphism.

Kernel of a Homomorphism:

If ϕ is a homomorphism from group G to $\bar{G}$,

then $\quad Ker(\phi) = \{x \in G \mid \phi(x) = e'\}$,

where e' is identity of $\bar{G}$.

Example: $\quad G = R^*$, $\quad \bar{G} = R^*$.

$\phi : G \to \bar{G}$

$\phi = I$ (Identity)

$$I(ab) = ab = I(a)\, I(b)$$

$\phi : R^* \to R^*$. $\quad \phi(x) = |x|$.

$$\phi(ab) = |ab| = |a|\,|b| = \phi(a)\,\phi(b) \quad \forall a, b \in G = R^*$$

$$\ker\{I\} = \{x \in R^* \mid I(x) = 1\}$$
$$= \{x \in R^* \mid x = 1\} = \{1\}$$

$$\ker\{\phi\} = \{x \in R^* \mid \phi(x) = 1\}$$
$$= \{x \in R^* \mid |x| = 1\} = \{-1, 1\}$$

Example!

Homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_n$.

$$\phi(k) = k \pmod n$$

$\boxed{\begin{array}{l} \mathbb{Z} \text{ to } \mathbb{Z}_{12} \\ \phi(k) = k \pmod{12} \\ \phi(15) = 3. \end{array}}$

$$\ker\{\phi\} = \{x \in G \mid \phi(x) = 0\}$$

$$= \{x \in G \mid x \pmod n = 0\}$$

$$= 0, \pm n, \pm 2n, \pm 3n, \ldots$$

$$= \langle n \rangle.$$

Properties of Homomorphism:

Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let $g \in G$. Then

① ϕ carries the identity of G to the identity of $\overline{G}$

i.e. $\phi(e) = e'$, where e' is identity of $\overline{G}$

② $\phi(g^n) = [\phi(g)]^n$

③ If $|g| = n$, then $|\phi(g)|$ divides n

④ If $\phi(g) = g'$, then $\phi^{-1}(g') = \{x \in G \mid \phi(x) = g'\}$
$= g \ker(\phi)$

$= g\,\mathrm{Ker}(\phi)$.

Proof:

① Let e be the identity of G and e' be the identity of $\bar{G}$.

$$e = ee$$

$$\Rightarrow \phi(e) = \phi(ee)$$

$$\Rightarrow \phi(e) = \phi(e)\,\phi(e)$$ [∵ ϕ is homomorphism $\phi(ab) = \phi(a)\phi(b)$]

R^*, $(2)\left(\frac{1}{2}\right) = \frac{1}{0}$

$$\Rightarrow \underline{\{\phi(e)\}^{-1}\phi(e)} = \underline{\{\phi(e)\}^{-1}\phi(e)}\,\phi(e)$$ [$\phi(e) \in \bar{G}$ & $\bar{G}$ is a group]

$$\Rightarrow e' = e' \cdot \phi(e)$$

$$\Rightarrow e' = \phi(e)$$

$$\Rightarrow \phi(e) = e'$$

$\Rightarrow$ ϕ carries identity of G to identity of $\bar{G}$

② Given that $g \in G$, TS $\phi(g^n) = [\phi(g)]^n$.

Case-I: for $n = 0$,

$$\phi(g^0) = \phi(e) = e' = [\phi(g)]^0$$

Case-II: for $n > 0$,

Case-II: for $n > 0$,

$$\phi(g^n) = \phi(g \cdot g \cdot g \cdots g)$$
$$= \phi(g) \cdot \phi(g) \cdots \phi(g)$$
$$= [\phi(g)]^n.$$

Case-III: for $n < 0$, $-n > 0$.

$$g^n g^{-n} = e$$

$$\Rightarrow \phi(g^n g^{-n}) = \phi(e)$$

$$\Rightarrow \phi(g^n)\,\phi(g^{-n}) = e'$$

$$\Rightarrow \phi(g^n) \cdot [\phi(g)]^{-n} = e' \quad \left[\because -n > 0 \text{ \& Case II}\right]$$

$$\Rightarrow \phi(g^n)\underbrace{[\phi(g)]^{-n}[\phi(g)]^{n}} = e'[\phi(g)]^n$$

$$\Rightarrow \phi(g^n) = [\phi(g)]^n.$$

(3) $|g| = n \Rightarrow g^n = e$

$$\Rightarrow \phi(g^n) = \phi(e)$$

$$\Rightarrow [\phi(g)]^n = e'$$

$\Rightarrow |\phi(g)|$ finites n.

④ Given that $\phi(g) = g'$

$\underline{\underline{TS}}$ $\phi^{-1}(g') = g \ker \phi$.

let $x \in \phi^{-1}(g') \Rightarrow \phi(x) = g'$

$\Rightarrow \phi(x) = \phi(g)$ $[\because \phi(g) = g']$

$\Rightarrow [\phi(g)]^{-1} \phi(x) = [\phi(g)]^{-1} \phi(g)$

$\Rightarrow \phi(g^{-1})\phi(x) = e'$

$\Rightarrow \phi(g^{-1}x) = e'$

$\Rightarrow g^{-1}x \in \ker(\phi)$

$\Rightarrow x \in g\ker(\phi)$

$\therefore \phi^{-1}(g') \subseteq g\ker\phi$. —— ①

Now, let $a \in \ker\phi \Rightarrow \phi(a) = e'$ —— ②

$a \in \ker(\phi) \Rightarrow ga \in g\ker\phi$

Now, $\phi(ga) = \phi(g)\phi(a) = \phi(g)e'$ (from ②)

$\Rightarrow \phi(ga) = \phi(g)e'$ (Given $\phi(g) = g'$)

$\Rightarrow \phi(ga) = g'e' = g'$

$\Rightarrow \quad ga \in \phi^{-1}(g')$

$\therefore \; g\ker(\phi) \subseteq \phi^{-1}(g')$ ——— (3)

from eq^n (2) & (3),

$\phi^{-1}(g') = g\ker\phi$.