(7.5) Least-Squares Solutions For Inconsistent Systems

In this section, we find an approximate solution to the system $Ax = b$, even if it is inconsistent.
i.e find a vector v s.t Av is close to b as possible.

Definition Least-Squares Solution

Let $Ax = b$ be a system of linear Equations, where A is $m \times n$ matrix and $b \in \mathbb{R}^m$. A vector $v \in \mathbb{R}^n$ is said to be Least-square solution to the system $Ax = b$ if the following condition is satisfied:

$$\|Av - b\| \leq \|Az - b\| \quad \text{for all } z \in \mathbb{R}^n$$

In other words, v is least-square solution to the system $Ax = b$ if Av is closest vector in $\mathbb{R}^m$ to b.

Theorem (7.14) Let $Ax = b$ be system of linear equations where A is $m \times n$ matrix and $b \in \mathbb{R}^m$. Let W be the subspace of $\mathbb{R}^m$ given by $W = \{Ax ; x \in \mathbb{R}^n\}$. Let $v \in \mathbb{R}^n$, then the following are Equivalent:

ⓐ v is least-square solution to the system $Ax = b$

ⓑ v satisfies $(A^T A) v = A^T b$

ⓒ v satisfies $Av = \text{proj}_W b$

Note: Least-Square solution to the system $Ax = b$ can be found by solving linear system

$$(A^T A) x = A^T b.$$

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Example (19): Find least square solution for the linear system $Ax=b$, where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

(Solution) To find least - square solution for system $Ax=b$, we need to solve

$$(A^T A)\, x = A^T b.$$

Now, $$A^T A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

thus, augmented matrix for system $(A^T A)\, x = A^T b$ is

$$\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right]$$

Now, row reduce augmented matrix,

$$\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{21} R_1} \left[\begin{array}{cc|c} 1 & 3/7 & 26/21 \\ 9 & 11 & 19 \end{array}\right]$$

after, applying row-reduction, we get

$$\left[\begin{array}{cc|c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$$

thus, we have

$$x = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} = \begin{bmatrix} 0.77 \\ 1.1 \end{bmatrix}$$ is desired least square solution.

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Note ①, we cross-check answers.

$$Av = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0.77 \\ 1.1 \end{bmatrix} = \begin{bmatrix} -0.33 \\ 4.84 \\ 4.18 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} = b$$

Av is close to b.

Note ② we can also check

$$\|Av - b\| \le \|Az - b\| \text{ for any } z \in \mathbb{R}^2.$$

For example, if $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$Az - b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \|Az - b\| = 1$$

and $Av - b = \begin{bmatrix} -0.33 \\ -0.16 \\ +0.18 \end{bmatrix} \Rightarrow \|Av - b\| = 0.4085$

$\therefore \|Av - b\| < \|Az - b\|$

Example Prove that least-square solution for the linear system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 4 & 1 \end{bmatrix} \; \& \; b = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

satisfies $\|Av - b\| \le \|Az - b\|$, where $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
(solution) [Try as Exercise]

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