Theory of Real Functions

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Day 1, 10/8/2020

Limits of functions

Definition Let $A \subseteq \mathbb{R}$, c be a cluster point of A. Let $f : A \to \mathbb{R}$ be a function. A real no. L is said to be limit of f at c i.e.

$$L = \lim_{x \to c} f(x) \quad \text{if given } \epsilon > 0$$

$\exists \; \delta > 0 \; \ni$ if $0 < |x - c| < \delta$, $x \in A$, then

$|f(x) - L| < \epsilon$.

first look at the very simple Example :-

Step function :- $[x]$ or greatest integer function

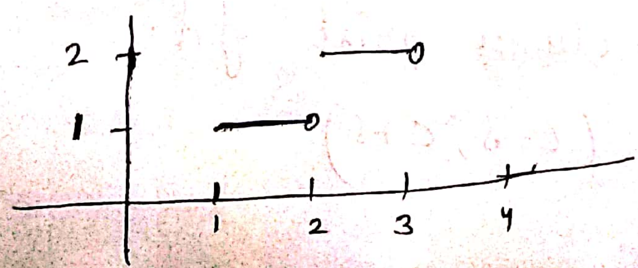
~~$[x] \leq x$~~

$f(x) = [x]$, this takes the value of greatest integer $\leq$

for Ex :- $f(1) = [1] = 1$

$f(1.1) = 1$, $f(3.67) = 3$, $f(0.9) = 0$

and so on



$f(2) = 2$

$$\lim_{x \to 2.75} f(x) = 2$$

Before looking at the '$\epsilon - \delta$' definition of limit of functions, look at the following :-

$$\lim_{n \to \infty} a_n = l$$

for $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ $\ni$ $|a_n - l| < \epsilon$ whenever $n \geq N$.

<u>Note 1</u> $0 < |x - c| < \delta$ is equivalent to $x \neq c$

<u>Note 2</u> if $\lim_{x \to c} f(x)$ does not exist, we say f diverges at c.

~~Ex:~~ ~~$f(x) = |x|$~~

~~Ques~~ ~~$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x|$?~~

~~Does it exist or not~~

<u>Cluster point of a set $A \subseteq \mathbb{R}$</u>

A point $c \in \mathbb{R}$ is a cluster point of A if for $\delta > 0$ $\exists$ at least one point $x \in A, x \neq c$, such that $|x - c| < \delta$.

In other words, if for every $\delta > 0$, $(c - \delta, c + \delta)$ contains a point of A other than c, then 'c' is said to be cluster point of A.

<u>Notation :-</u> $V_\delta(c) = (c - \delta, c + \delta)$

Example 1 Let $A = \{1, 2\}$

Question is :- Is '1' a cluster point of A?

let $\delta = 1/2$, $(1-\delta, 1+\delta) = (1/2, 3/2)$

Contains a point '1' of A, there is no point of A in $(1/2, 3/2)$ other than 1.

$\therefore$ 1 is not a cluster point of A.

$\|^{ly}$ 2 is not a cluster point of A.

Question Is there any point of $\mathbb{R}$ which is a cluster point of A? Answer is NO

Reason let pick up $c = 3$

then $(3-\delta, 3+\delta)$, for $\delta = 1/2$, is $(2\frac{1}{2}, 3.5)$

which does not contain any point of A.

$\|^{ly}$ for others point of $\mathbb{R}$.

$\Rightarrow$ A has no cluster points.

Theorem 4.1.2 A no. $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ iff there exists a sequence (a_n) in A such that $\lim_{n\to\infty} a_n = c$.

and $a_n \neq c \;\forall\; n \in \mathbb{N}$.

Proof. Let c be a cluster point of A $\therefore$ for every $\delta > 0$, $(c-\delta, c+\delta)$ contains a point of A other than c. In particular, for every $n \in \mathbb{N}$, $V_{1/n}(c)$ contains ~~at~~ at least one point $a_n \in A$ other than c. That is, $a_n \in (c - 1/n, c + 1/n)$

$$\Rightarrow \quad 0 < |a_n - c| < 1/n \quad [\because a_n \neq c)$$

$$\Rightarrow \quad \lim_{n \to \infty} a_n = c.$$

Conversely, Let a_n be a sequence in A other than c such that $\lim_{n\to\infty} a_n = c$.

Let $\delta > 0$ $\therefore$ $\exists$ $K \in \mathbb{N}$ $\ni$ $|a_n - c| < \delta$ $\forall n \geq K$.

$$\Rightarrow \quad a_n \in (c-\delta, c+\delta) \ \forall n \geq K$$

$\Rightarrow$ $(c-\delta, c+\delta)$ contains points $a_n \in A$, $a_n \neq c$, $\therefore$ c is a cluster point of A.

Example 4.1.3 (a) Let $A_1 = [0, 1)$

Ques: ~~Cluster points of A_1 is $[0,1]$~~

find out the cluster points of A_1?

We show that each point of $[0,1]$ is a cluster point of $(0,1) = A_1$.

Consider the point 0.1, if we look at

$V_\delta(0.1) = (0.1-\delta, 0.1+\delta)$ for any $\delta > 0$.

If we pick $\delta = 0.01$

then $(0.09, 0.11)$ contains infinitely many points of $(0,1)$ other than 0.1.

So 0.1 is cluster point of A_1

$||^{ly}$ any point of $(0,1)$ is a cluster point of A_1.

Now look at 0,

$(0-\delta, 0+\delta)$, Pick any $\delta > 0$,

One can see $(0-\delta, 0+\delta)$ contains points of $(0,1)$ other than 0.

$||^{ly}$ for 1.

Hence, collection of cluster points of $(0,1)$ is $[0,1]$.