

Inequalities

Result 1. The function $f(x) = \frac{x}{1+x}$, $x \geq 0$, is monotonically increasing.

Pf. Let $y > x \geq 0$. Then $1+y > 1+x \geq 1$ and so

$$\frac{1}{1+y} < \frac{1}{1+x} \Rightarrow 1 - \frac{1}{1+y} > 1 - \frac{1}{1+x}$$

$$\Rightarrow \frac{y}{1+y} > \frac{x}{1+x} \Rightarrow f(y) > f(x).$$

Result 2. For any $x, y \in \mathbb{R}$, we have

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Pf. Let x and y have the same sign; $x \geq 0, y \geq 0$ (w/o loss of generality). Therefore,

$$\begin{aligned} \frac{|x+y|}{1+|x+y|} &= \frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y} \\ &\leq \frac{x}{1+x} + \frac{y}{1+y} \\ &= \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}. \end{aligned}$$

Next let x and y have different signs. We may assume that $|x| \geq |y|$. Then $|x+y| \leq |x|$.

$$\text{In view of Result 1, } \frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

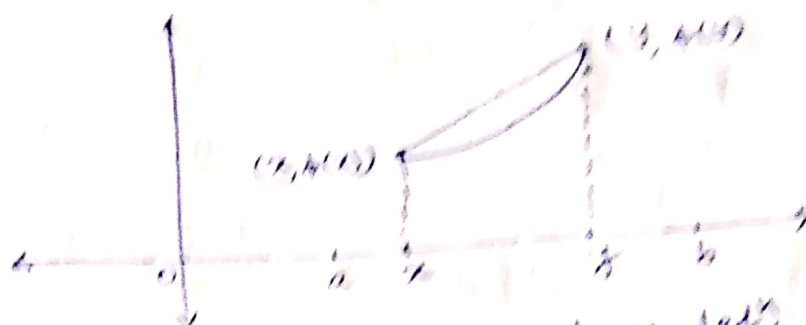
Result 3. "AM-GM Inequality"

If $a > 0$, $b > 0$ and $\lambda \in (0, 1)$ is fixed. Then,

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b.$$

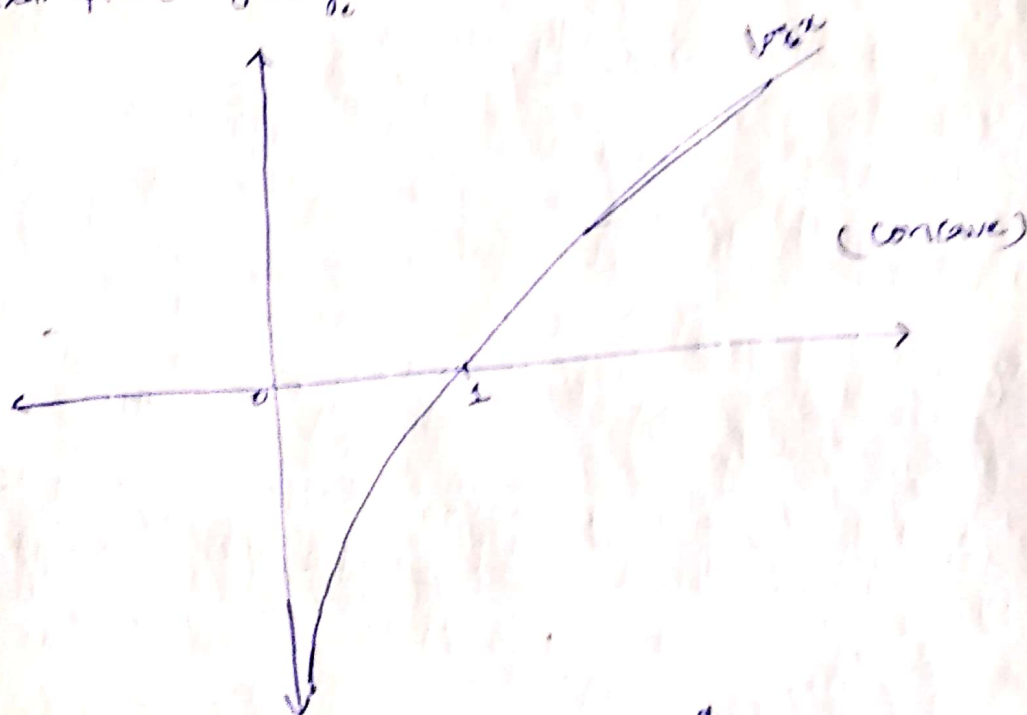
[Convex function: A function $\phi: (a, b) \rightarrow \mathbb{R}$ is said to be convex if for each $x, y \in (a, b)$ and each $\lambda \in [0, 1]$ we have

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda \phi(x) + (1-\lambda)\phi(y)$$

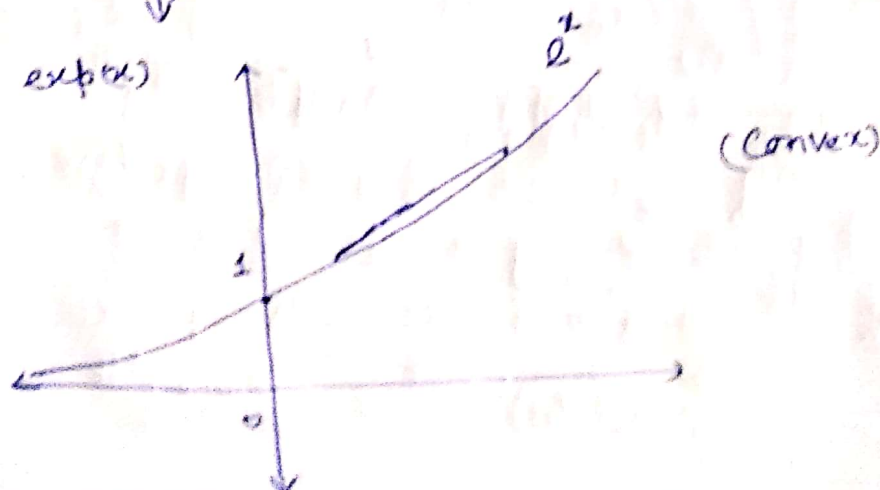


If the inequality in the above defⁿ is reversed, the function is said to be concave

example: (i) $y = \log_e x$ ($x > 0$)



(ii) $y = \exp(x)$



(Pfr of Result 3)

Since $y = \ln x$ ($x > 0$), is concave, we have

$$\ln(\lambda a + (1-\lambda)b) \geq \lambda \ln a + (1-\lambda) \ln b,$$

$$\text{i.e., } \ln a^\lambda b^{(1-\lambda)} \leq \ln(\lambda a + (1-\lambda)b).$$

As $y = \exp(x)$ is a strictly increasing function, it follows from the above inequality that

$$a^\lambda b^{(1-\lambda)} \leq \lambda a + (1-\lambda)b.$$

Alternative proof:

If $a=b$, then the inequality is obvious, in fact, in this case the inequality is actually an equality.

Let $a \neq b$ and $a < b$ (to be specific).

Consider $f(x) = x^{1-\lambda}$ defined on $[a, b]$. Then

$\exists c \in (a, b)$ such that

$$\frac{b^{1-\lambda} - a^{1-\lambda}}{b-a} = (1-\lambda)c^{-\lambda} \quad (\text{in view of})$$

Lagrange's Mean Value Theorem)

$$\Rightarrow b^{1-\lambda} - a^{1-\lambda} = (b-a)(1-\lambda)c^{-\lambda} < (b-a)(1-\lambda)a^{-\lambda}$$

(as $a < c < b \Rightarrow a^\lambda < c^\lambda < b^\lambda$)

on multiplying throughout with $a^{1-\lambda} (> 0)$ we get

$$a^\lambda b^{(1-\lambda)} - a < (b-a)(1-\lambda) = b-a-\lambda(b-a)$$

$$\Rightarrow a^\lambda b^{(1-\lambda)} < \lambda a + (1-\lambda)b.$$

Remark: When $x \geq 0$, $y \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q.$$

The inequality is obvious if either $x=0$ or $y=0$.

Let $x \neq 0 \neq y$. Then $x > 0$ and $y > 0$.

put $a = x^p$, $b = y^q$, $\lambda = \frac{1}{p}$ in AM-GM inequality we have

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q \quad (\text{as } \frac{1}{p} + \frac{1}{q} = 1)$$

Result 4 (Hölder's Inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \dots, n$ and suppose that $p \geq 1$ and $q \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q}$$

In particular, when $p = q = 2$, the inequality reduces to

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}$$

(Cauchy-Schwarz Inequality)

pf. If either $\sum_{i=1}^n x_i^p = 0$ or $\sum_{i=1}^n y_i^q = 0$, then

inequality is clear. Let $\sum_{i=1}^n x_i^p \neq 0 \neq \sum_{i=1}^n y_i^q$.

Put $\alpha_i = \frac{x_i}{\left(\sum_{i=1}^n x_i^p \right)^{1/p}}$ and $\beta_i = \frac{y_i}{\left(\sum_{i=1}^n y_i^q \right)^{1/q}} : 1 \leq i \leq n$

Then by a consequence of AM-GM inequality,

we have
$$\alpha_i \beta_i \leq \frac{1}{p} \alpha_i^p + \frac{1}{q} \beta_i^q \quad (1 \leq i \leq n)$$

$$\Rightarrow \frac{x_i y_i}{\left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q}} \leq \frac{1}{p} \frac{x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \frac{y_i^q}{\sum_{i=1}^n y_i^q}$$

Taking summation over i , we get

$$\frac{\sum_{i=1}^n x_i y_i}{\left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q}} \leq \frac{1}{p} \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \frac{\sum_{i=1}^n y_i^q}{\sum_{i=1}^n y_i^q} = \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q}$$

Result 5 (Minkowski's Inequality)

Let $x_i \geq 0$ and $y_i \geq 0$ ($1 \leq i \leq n$) and suppose that $p \geq 1$. Then

$$\left(\sum_{i=1}^n (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} + \left(\sum_{i=1}^n y_i^p\right)^{1/p}$$

pf If $p=1$, the inequality is self-evident.

Let $p > 1$. Suppose $q > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1. \quad \left(\begin{array}{l} \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \\ \Rightarrow p = q(p-1) \end{array} \right)$$

we have

$$\begin{aligned} \sum_{i=1}^n (x_i + y_i)^p &= \sum_{i=1}^n (x_i + y_i) (x_i + y_i)^{p-1} \\ &= \sum_{i=1}^n x_i (x_i + y_i)^{p-1} + \sum_{i=1}^n y_i (x_i + y_i)^{p-1} \\ &\leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n (x_i + y_i)^{(p-1)q}\right)^{1/q} + \\ &\quad \left(\sum_{i=1}^n y_i^q\right)^{1/q} \left(\sum_{i=1}^n (x_i + y_i)^{(p-1)p}\right)^{1/p} \end{aligned}$$

(In view of Hölder's Inequality)

Thus

$$\sum_{i=1}^n (x_i + y_i)^p \leq \left[\left(\sum_{i=1}^n x_i^p \right)^{1/p} + \left(\sum_{i=1}^n y_i^p \right)^{1/p} \right] \left(\sum_{i=1}^n (x_i + y_i)^p \right)^{1/q} \quad (6)$$

$$\Rightarrow \left(\sum_{i=1}^n (x_i + y_i)^p \right)^{1-1/q} \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} + \left(\sum_{i=1}^n y_i^p \right)^{1/p}$$

(provided $\sum_{i=1}^n (x_i + y_i)^p \neq 0$, otherwise inequality is obvious)

$$\Rightarrow \left(\sum_{i=1}^n (x_i + y_i)^p \right)^{1/p} \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} + \left(\sum_{i=1}^n y_i^p \right)^{1/p}$$

Result 6 (Minkowski's Inequality for Infinite Sums)

Let $p \geq 1$ and $\{x_n\}_{n \in \mathbb{N}}$, $\{y_n\}_{n \in \mathbb{N}}$ be sequences of non-negative terms such that $\sum_{n \in \mathbb{N}} x_n^p$ and $\sum_{n \in \mathbb{N}} y_n^p$ are convergent. Then $\sum_{n \in \mathbb{N}} (x_n + y_n)^p$ is cgt. and

$$\left(\sum_{n=1}^{\infty} (x_n + y_n)^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p \right)^{1/p}$$

Pf. For any $m \in \mathbb{N}$, we have from Hölder's Inequality,

$$\begin{aligned} \left(\sum_{n=1}^m (x_n + y_n)^p \right)^{1/p} &\leq \left(\sum_{n=1}^m x_n^p \right)^{1/p} + \left(\sum_{n=1}^m y_n^p \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p \right)^{1/p} \end{aligned}$$

Thus $\left\{ \left(\sum_{n=1}^m (x_n + y_n)^p \right)^{1/p} \right\}_{m \in \mathbb{N}}$ is an monotonically

increasing sequence of non-negative real numbers bounded above by a fixed real number $\left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p \right)^{1/p}$. Hence it will converge to its l.u.b. Therefore,

$$\left(\sum_{n=1}^{\infty} (x_n + y_n)^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p \right)^{1/p}$$