Special Types of first-order Equations :-

(a) Equations Involving Only p and q :-

for equation of the type

$$f(p, q) = 0 \quad \text{--- (1)}$$

Charpit's eq^n reduce to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} = \frac{dp}{0} = \frac{dq}{0}$$

or from last two fractions of, we get

$$dp = 0$$

or $\boxed{p = a}$ --- (2)

where a is constant.

from eq^n (1) $f(a, q) = 0$ --- (3)

or

$q = Q(a)$ --- (4) a Constant

the sol^n of eq^n (1) is

$$z = ax + Q(a)y + b,$$

where b is a constant.

Example :- Solns

(1) $pq = 1$

Soln

$p = a, \quad q = 1/a$

$$\boxed{z = ax + \frac{1}{a}y + b}$$

where $a, \& b$ are constant.

(b) Equations Not Involving the independent Variables :-

If the partial differential equation is of the type

$$f(z, p, q) = 0 \qquad \text{——(1)}$$

then Charpit's eqn take the form

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z} \qquad \text{——(2)}$$

Taking last two functions of Charpit's eqn

$$\frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

or

$$\frac{dp}{p} = \frac{dq}{q}$$

on Integration

$$\boxed{p = aq} \quad \text{—— (3)}$$

Example :- Find a complete integral of the equation $p^2z^2 + q^2 = 1$. —— (1)

Since equation involve p, q & z only

putting $p = aq$ in eqn (1), we get

$$q = (1+a^2z^2)^{-1/2}$$

&

$$p = a(1+a^2z^2)^{-1/2}$$

then

$$dz = p\,dx + q\,dy$$

$$dz = a(1+a^2z^2)^{-1/2}dx + (1+a^2z^2)^{-1/2}dy$$

~~a+a~~

$$(1+a^2z^2)^{1/2}\,dz = a\,dx + dy$$

$$az(1+a^2z^2)^{1/2} - \log\left[az + (1+a^2z^2)^{1/2}\right]$$

$$= 2a(ax + y + b)$$

where a & b are constants.

<u>Example ②</u> $z = p^2 - q^2$

(5)

(C) Separable Equations:-

If given eqn can be written in the form

$$f(x,p) = g(y,q) \quad \text{--- (1)}$$

then Charpit's eqn become

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y} \quad \text{--- (2)}$$

Taking ~~first two~~ first & 4th functions, we get following ordinary differential eqn

$$\frac{dp}{dx} + \frac{f_x}{f_p} = 0$$

or

$$f_p\, dp + f_x\, dx = 0$$

$$df(x,p) = 0$$

or $\boxed{f(x,p) = a}$

Similarly by taking 2nd & 5th function we get

$$\boxed{g(y,q) = a}$$

Example 6 find a complete integral of the eqn

$$p^2 y(1+x^2) = q x^2 \quad \text{--- (1)}$$

We can rewrite eqn (1) as

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a^2 \text{ (say)}$$

(Constant)

So that

$$p = \frac{ax}{\sqrt{1+x^2}}, \quad \& \quad q = a^2 y$$

Hence

$$dz = p\,dx + q\,dy$$

$$dz = \frac{ax}{\sqrt{1+x^2}}\,dx + a^2 y\,dy$$

By Integration

$$\boxed{z = a\sqrt{1+x^2} + \frac{1}{2}a^2 y^2 + b}$$

When a & b are constant.

Exercise (1) Solve $p^2 q(x^2+y^2) = p^2 + q$

(d) Clairaut Equations :-

A first order partial differential eq^n is said to be of clairaut type if it can be written in the form

~~z = pdx +~~

$$\boxed{z = px + qy + f(p,q)} \quad \text{— (I)}$$

Charpit's eq^{ns} are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

So that, we get

$$p = a, \quad \& \quad q = b$$

then eq^n (1) become

$$\boxed{z = ax + by + f(a,b)} \quad \text{— (2)}$$

which is a sol^n of eq^n (I).

Example :- Find a complete integral of the equation

$$(p+q)(z-xp-yq)=1 \quad \text{--- (1)}$$

or

$$z = xp + yq + \frac{1}{p+q} \quad \text{--- (2)}$$

which is in clairaut's form

hence complete integral of (1)

Exercise (Find complete integral of following eqns :

(1) $zpq = p+q$

(2) $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

(3) $pqz = p^2(xq+p^2) + q^2(yp+q^2)$