In view of condition *(iii)* *i.e.* $u = 0$ at $y = 0$ for $0 \le x \le l$; we have

$$B_r + D_r = 0$$

or
$$D_r = -B_r \qquad \text{... (10)}$$

$$\therefore \quad u = \sum_{r=1}^{\infty} B_r \left(e^{r\pi y/l} - e^{-r\pi y/l}\right) \sin \frac{r\pi x}{l} = \sum_{r=1}^{\infty} 2B_r \sinh \frac{r\pi y}{l} \sin \frac{r\pi x}{l}$$

$$= \sum_{r=1}^{\infty} N_r \sinh \frac{r\pi y}{l} \sin \frac{r\pi x}{l} \qquad \text{... (11)}$$

(setting $2B_r = N_r$)

Using *(iv)* boundary condition, equation (11) yields

$$F(x) = \sum_{r=1}^{\infty} N_r \sinh \frac{r\pi b}{l} \sin \frac{r\pi x}{l} \qquad \text{... (12)}$$

This equation represents half-range Fourier series with the general constant $N_r \sinh \frac{r\pi b}{l}$ which may be determind by usual method of Fourier series.

Multiplying both sides of (12) by $\sin \frac{m\pi x}{l}$ and integrating between the limits $(0, l)$, we get

$$\int_0^l F(x) \sin \frac{m\pi x}{l}\, dx = \sum_{r=1}^{\infty} N_r \sinh \frac{r\pi b}{l} \int_0^l \sin \frac{r\pi x}{r} \sin \frac{m\pi x}{l}\, dx$$

$$= \sum_{r=1}^{\infty} N_r \sinh \frac{r\pi b}{l} \frac{l}{2} \delta_{r,m} = N_m \sinh \frac{m\pi b}{l} \cdot \frac{l}{2}$$

i.e.
$$N_m = \frac{1}{\sinh \frac{m\pi b}{l}} \cdot \frac{2}{l} \int_0^l F(x) \sin \frac{m\pi x}{l}\, dx$$

or
$$N_r = \frac{1}{\sinh \frac{r\pi b}{l}} \frac{2}{l} \int_0^l F(t) \sin \frac{r\pi t}{l}\, dt \qquad \text{... (13)}$$

Substituting this value of N_r in equation (12), we get the required solution of temperature distribution :

$$u = \frac{2}{l} \sum_{r=1}^{\infty} \frac{1}{\sinh \frac{r\pi b}{l}} \sinh \frac{r\pi y}{l} \left[\int_a^l F(t) \sin \frac{r\pi t}{l}\, dt \right] \sin \frac{r\pi x}{r} \qquad \text{... (14)}$$

9·4. Solution of Laplace's equation in Two-dimensional Cylindrical coordinates. (r, θ) : Circular Harmonics

The *Laplace equation* $\nabla^2 u = 0$ in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{... (1)}$$

If we assume that the function u is independent of coordinate z, then *Laplace's equation in two-dimensional* cylindrical coordinates takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{... (2)}$$

In this equation u is the function of r and θ ; therefore by the method of separation of variables u may be written as

$$u(r, \theta) = R(r)\,\Theta(\theta) \quad \text{... (3)}$$

where R is the function of r only and Θ is the function of θ only. Substituting this value of u in equation (2) ; we have

$$\frac{\Theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} = 0$$

Dividing this equation by $\frac{R\Theta}{r^2}$, we get

$$\frac{1}{R}\left[r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r}\right] = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \quad \text{... (4)}$$

In this equation left hand side is the function of r only ; while right hand side is the function of θ only ; therefore each side must be equal to the same constant n^2 (say).

i.e.

$$\frac{1}{R}\left[r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r}\right] = n^2$$

or

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - n^2 R = 0 \quad \text{... (5)}$$

and

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = n^2$$

or

$$\frac{\partial^2\Theta}{\partial\theta^2} + n^2\Theta = 0 \quad \text{... (6)}$$

The solution of equation (5) may be expressed as

$$R = A_n r^n + B_n r^{-n},\ n \neq 0 \quad \text{... (7)}$$

while equation (6) represents simple harmonic motion and its solution may be expressed as

$$\Theta = C_n \cos n\theta + D_n \sin n\theta,\ n \neq 0 \quad \text{... (8)}$$

where A_n, B_n, C_n are arbitrary constants.

If $n = 0$, the equations (5) and (6) take the form

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} = 0 \quad \text{... (9)}$$

and

$$\frac{\partial^2\Theta}{\partial\theta^2} = 0 \quad \text{... (10)}$$

The solution of these equations may be expressed as

$$R = A_0 \log_e r + B_0 \quad \text{... (11)}$$

and

$$\Theta = C_0\theta + D_0 \quad \text{... (12)}$$

The solution of Laplace's equation in cylindrical coordinates when the function u is independent of z are called *circular harmonics* and the number n is called the degree of harmonic. The circular harmonics $u_0(r, \theta)$ and $u_n(r, \theta)$ of degree zero and n are respectively given by

$$u_0(r, \theta) = (A_0 \log r + B_0)(C_0\theta + D_0) \quad \text{... (13)}$$

$$u_n(r, \theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \quad \text{... (14)}$$

A general single valued solution of Laplace's equation may be obtained by summing up the solutions (13) and (14) for all integral values of n and thus we have

$$u = a_0 \log_e r + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) + \sum_{n=1}^{\infty} r^{-n}(c_n \cos n\theta + d_n \sin n\theta) + c_0 \quad \text{... (15)}$$

where a_0, a_n b_n, c_n, d_n and c_0 are arbitrary constants.

$$a_n = \frac{1}{\pi R^n}\left[\int_0^{\pi} T_1 \cos n\theta \, d\theta + \int_{\pi}^{2\pi} T_2 \cos n\theta \, d\theta\right] = 0 \qquad \text{... (11)}$$

and $$b_n = \frac{1}{\pi R^n}\left[\int_0^{\pi} T_1 \sin n\theta \, d\theta + \int_{\pi}^{2\pi} T_2 \sin n\theta \, d\theta\right]$$

$$= \frac{1}{\pi R^n}\left[\frac{T_1 - T_2}{n}(1 - \cos n\pi)\right] = \begin{cases} \dfrac{2}{n\pi R^n} \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even} \end{cases} \qquad \text{... (12)}$$

Substituting values of c_0, a_n and b_n in equation (5), the required steady state temperature in the region inside the cylinder is given by

$$u = \frac{T_1 + T_2}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}(T_1 - T_2) \sin n\theta \left(\frac{r}{R}\right)^n ; n = 1, 3, 5... \qquad \text{... (13)}$$

$$u = \frac{T_1 + T_2}{2} + \frac{2(T_1 - T_2)}{\pi} \sum_{p=1}^{\infty} \left(\frac{r}{R}\right)^{2p-1} \frac{\sin(2p-1)\theta}{2p-1}$$

$$(p = 1, 2, 3 \ldots) \qquad \text{... (14)}$$

9.5. Solution of Laplace's Equation in General Cylindrical coordinates (General Cylindrical Harmonics)

The Laplace's equation

$$\nabla^2 u = 0$$

in cylindrical coordinates (r, θ, z) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{... (1)}$$

In this equation u is the function of r, θ and z, therefore by the method of separation of variables u may be written as $u(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z)$ ... (2)

where R is the function of r only, Θ is the function of θ only and Z is the function of z only. Substituting value of u in equation (1) and dividing by $R\,\Theta\,Z$, we get

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

or $$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{1}{r^2\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} \qquad \text{... (3)}$$

In this equation left hand side is the function of z alone while right hand side is the function of r and θ and is independent of z, therefore each side must be equal to the same constant k^2 (say), *i.e.*

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 \text{ or } \frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0 \qquad \text{... (4)}$$

and $$-\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{1}{r^2\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = k^2 \qquad \text{... (5)}$$

Multiplying equation (5) throughout by r^2, we get

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \frac{r}{R}\frac{\partial R}{\partial r} - \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = k^2 r^2 \quad \text{or} \quad \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + k^2 r^2) = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} \qquad \text{... (6)}$$

Again the left hand side is the function of r only ; while right hand side is the function of θ only ; therefore each side must be equal to the same constant m^2 (say), *i.e.*

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = m^2 \text{ or } \frac{\partial^2\Theta}{\partial\theta^2} + m^2\Theta = 0 \qquad \dots (7)$$

or

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + k^2 r^2 = m^2$$

or

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (k^2 r^2 - m^2) R = 0 \qquad \dots (8)$$

The solution of equation (4) is given as

$$Z = C_1 e^{kz} + C_2 e^{-kz} \qquad \dots (9)$$

The solution of equation (7) may be expressed as

$$\Theta = C_3 \cos m\theta + C_4 \sin m\theta \qquad \dots (10)$$

In order to solve equation (8), let us substitute

$$kr = x \qquad \dots (11)$$

then equation (8) takes the form

$$x^2\frac{\partial^2 R}{\partial x^2} + x\frac{\partial R}{\partial x} + (x^2 - m^2) R = 0 \qquad \dots (12)$$

This is *Bessel's differential equation.* Its solution is given by

or $$R = C_5 J_m (x) + C_6 J_{-m} (x) = C_5 J_m (kr) + C_6 J_{-m} (kr) \qquad \dots (13a)$$

for m as a fraction

or $$R = C_5 J_m (kr) + C_6 Y_m (kr) \qquad \dots (13b)$$

for m an integer or in general

Thus the general solution of Laplace's equation in cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned} u(r, \theta\, z) &= R(r)\,\Theta(\theta)\,Z(z) \\ &= [C_1 e^{kz} + C_2 e^{-kz}]\,[C_3 \cos m\theta + C_4 \sin m\theta]\,[C_5 J_m (kr) + C_6 Y_m (kr)] \end{aligned} \qquad \dots (14)$$

These solutions of Laplace's equation are called general *cylindrical harmonics*. If we let k be a fixed constant and if we require u to be a single valued function of θ, then m must take only integral values and consequently the solution takes the form

$$u = \sum_{m=0}^{\infty} e^{kz} (A_m \cos m\theta + B_m \sin m\theta) + \left[e^{-kz} (C_m \cos m\theta + D_m \sin m\theta)\right] J_m (kr) \qquad \dots (15)$$

This solution remains finite at $r = 0$ and is specially useful in certain electrical problems and the problems of steady state heat condition. The constants in the solution may be evaluated by using the boundary conditions of the specified problem.

9·6 Solution of Laplace's Equation in Spherical Polar Coordinates : Spherical Harmonics

The Laplace's equation

$$\nabla^2 u = 0$$

in spherical polar coordinates (r, θ, ϕ) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \qquad \dots (1)$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

From this equation it is obvious that u is the function of (r, θ, ϕ) ; therefore by the method of separation of variables its solution may be expressed as

$$u(r, \theta, \phi) = R\,\Theta\Phi \quad \text{... (2)}$$

where R is function of r only, Θ is the function of θ only and Φ is the function of ϕ only.

Substituting (2) in (1), we get

$$\frac{\partial}{\partial r}\left\{r^2\frac{\partial(R\Theta\Phi)}{\partial r}\right\}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial(R\,\Theta\Phi)}{\partial\theta}\right\}+\frac{1}{\sin^2\theta}\frac{\partial^2(R\Theta\Phi)}{\partial\phi^2}=0$$

Dividing by $R\Theta\Phi$, we get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Multiplying throughout by $\sin^2\theta$, we get

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\Phi^2}=0$$

$$\Rightarrow\quad \frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \quad \text{...(3)}$$

In this equation LHS is the function of r and θ only while RHS is a function of ϕ only, therefore for the validity of this equation, each side must be equal to same constant m^2 (say), so that

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=m^2 \quad \text{...(4)}$$

and $$-\frac{1}{\phi}\frac{\partial^2\Phi}{\partial\phi^2}=m^2 \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial\phi^2}+m^2\Phi=0 \quad \text{...(5)}$$

Now dividing equation (4) by $\sin^2\theta$, we get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=\frac{m^2}{\sin^2\theta}$$

$$\Rightarrow\quad \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{m^2}{\sin^2\theta}$$

In this equation LHS is the function of r only while RHS is the function of θ only; therefore for the validity of this equation, each side must be equal to same constant $n(n+1)$ say.

$$\therefore\quad \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=n(n+1) \quad\Rightarrow\quad r^2\frac{\partial^2 R}{\partial r^2}+2r\frac{\partial R}{\partial r}-n(n+1)\,R=0 \quad \text{...(6)}$$

and $$-\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{m^2}{\sin^2\theta}=n(n+1)$$

$$\Rightarrow\quad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \quad \text{...(7)}$$

The solution of equation (6) is

$$R = A\,r^n + B r^{-n-1} \quad \text{...(8)}$$

In equation (7), we put $x = \cos\theta$ and transform the variable θ in x.

$\dfrac{\partial\Theta}{\partial\theta}=\dfrac{\partial\Theta}{\partial x}\cdot\dfrac{\partial x}{\partial\theta}=-\sin\theta\dfrac{\partial\Theta}{\partial x}$, so equation (7) becomes

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\frac{\partial\Theta}{\partial x}\right]+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial \Theta}{\partial x}\right] + \left[n\,(n+1) - \frac{m^2}{(1-x^2)}\right]\Theta = 0$$

$$\Rightarrow \quad (1-x^2)\frac{\partial^2 \Theta}{\partial x^2} - 2x\frac{\partial \Theta}{\partial x} + \left[n\,(n+1) - \frac{m^2}{(1-x^2)}\right]\Theta = 0 \qquad \ldots(9)$$

This is associated Lagendre's equation, its solution is

$$\Theta = C\,P_n^m(x) + D\,Q_n^m(x)$$

or $\quad \Theta = C\,P_n^m(\cos\theta) + D\,Q_n^m(\cos\theta) \qquad \ldots(10)$

The solution of equation (5) is

$$\phi = E\cos(m\phi) + F\sin(m\phi) \qquad \ldots(11)$$

Therefore the solution of Laplace's equation will be

$$u = R\Theta\,\Phi = (Ar^n + Br^{-n-1})\,[CP_n^m(\cos\theta) + D\,Q_n^m(\cos\theta)]\,(E\cos m\,\phi + F\sin m\,\phi) \ldots(12)$$

The most general solution of Laplace's equation will be

$$u = \sum_n \sum_m (A_n r^n + B_n r^{-n-1})\,\{C_{mn}P_n^m(\cos\theta) + D_{mn}\,Q_n^m(\cos\theta)\,(E_m\cos m\phi + F_m\sin m\phi) \qquad \ldots(13)$$

Simpler Solution

The solution is too complicated to handle, therefore usually angular part of solution is handled together, for this we put

$$S(\theta,\phi) = \Theta(\theta)\,\Phi(\phi) \qquad \ldots(14)$$

$S(\theta, \phi)$ is the function of θ and ϕ and is called the *surface harmonic.* If ϕ is constant, then S is the function of θ only and is called the surface *zonal harmonic.* For this we put

$$u(r,\theta,\phi) = R(r)\;S(\theta,\phi) \qquad \ldots(15)$$

Substituting $u = RS$ from (2) in (1) and dividing by RS ; we get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{S\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} = 0$$

or $$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) - \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} \qquad \ldots(16)$$

In this equation the left hand side is the function of r only and right hand side is the function of θ and ϕ ; therefore each side must be equal to the same constant ; $n\,(n+1)$ say ; n being a constant

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = n\,(n+1)$$

i.e. $$r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n\,(n+1)\,R = 0 \qquad \ldots(17)$$

and $$-\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) - \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} = n\,(n+1)$$

or $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} + n\,(n+1)\,S = 0 \qquad \ldots(18)$$

The solution of equation (17) is

$$R = Ar^n + B\,r^{-n-1} \qquad \ldots(19)$$

If $S = S_n$ is the solution of equation (18), then the solution of Laplace's equation in spherical polar coordinates is expressed as

$$u = RS = (A\,r^n + B\,r^{-n-1})\,S_n \qquad \ldots(20)$$

This solution is called the *spherical harmonic*. The subscript n on S_n signifies that the same value of n must be used in both terms of equation (8).

Surface Zonal Harmonics. If the function u is independent of ϕ, then S_n is the function of θ only *i.e.* S_n is the zonal surface harmonic. In this case we have

$$\frac{\partial^2 S}{\partial \phi^2} = 0 \quad \ldots (21)$$

and equation (18) reduces to

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + n(n+1)S_n = 0 \quad \ldots (22)$$

In we substitute $x = \cos\theta$ and transfrom the independent variable θ to the independent variable x

$$\frac{\partial S_n}{\partial\theta} = -\sin\theta\frac{\partial S_n}{\partial x}$$

in equation (22) ; we obtain

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{(1-x^2)\frac{\partial S_n}{\partial x}\right\} + n(n+1)S_n = 0$$

or
$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial S_n}{\partial x}\right\} + n(n+1)S_n = 0 \quad \ldots (23)$$

This equation is *Legendre differential equation.* If n is positive integer, the solution of equation (23) is *Legendre polynomial* given by

$$S_n = P_n(x) = P_n(\cos\theta) \quad \ldots (24)$$

As Laplace's equation is linear equation, the linear combination of solutions of the type of (20) is also a solution. Hence the general solution of Laplace's equation when S_n is independent of ϕ is given by

$$u = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta) \quad \ldots (25)$$

where A_n and B_n are constants to be determined by boundary conditions of specified problem.

SOLVED EXAMPLES

Ex. 8. *If S_n and S_m are zonal spherical harmonics, then prove that*

$$\iint_S S_n S_m \, dS = 0, \; n \neq m$$

(*Kanpur 1998,Meerut 1997*)

Solution. The Green's theorem is

$$\iiint_V (\Psi_1\nabla^2\Psi_2 - \Psi_2\nabla^2\Psi_1)\, dV = \iint_S (\Psi_1\nabla\Psi_2 - \Psi_2\nabla\Psi_1)\, dS \quad \ldots (1)$$

Let us substitute

$$\Psi_1 = r^m S_m \text{ and } \Psi_2 = r^n S_n \quad \ldots (2)$$

then Ψ_1 and Ψ_2 satisfy Laplace's equation, *i.e.*

$$\nabla^2\Psi_1 = 0 \text{ and } \nabla^2\Psi_2 = 0.$$

Hence the volume integral on left hand side of (1) vanishes. If we consider the *surface of a unit sphere in Green's theorem* ; then we have

$$(\nabla\Psi_1)_s = \frac{\partial}{\partial r}(r^m S_m) = mr^{m-1}S_m = mS_m \text{ at } r = 1$$

and
$$\frac{1}{h^2\,\tau}\frac{\partial \tau}{\partial t} = -k^2 \quad \text{or} \quad \frac{\partial \tau}{\partial t} = -k^2 h^2\,\tau \qquad \text{... (5)}$$

Equation (4) is the *Helmholtz equation* which may again be solved by the method of separation of variables. Equation (5) may be expressed as

$$\frac{\delta \tau}{\tau} = -k^2 h^2\,\delta t$$

Integrating, we get

$$\log \tau = -k^2 h^2\,t + \log A \qquad \text{... (6)}$$

where $\log A$ is constant of integrtion. Equation (6) gives

$$\tau = A\,e^{-k^2h^2t} \qquad \text{... (7)}$$

Thus the complete solution of equation (1) is

$$u = \phi(x, y, z)\,\tau(t) = \phi(x, y, z)e^{-k^2h^2t}$$

The reason for choosing the constant $(-k^2)$ to be negative is that the temperature of a body decreases with increase of time. Now we shall discuss the solution of heat flow equation in different dimensions under certain boundary conditions.

Variable Linear Flow (Heat Flow Equation in one dimension)

Let us consider a bar of finite length l and of uniform section, the diameter of which is small in comparison with the radius of curvature. Let us assume that *the surface is impervious to heat so that there is no loss of radiation from the sides.* Let the initial temperature of the bar be given and its ends be kept at constant temperature zero. If one end of the bar is fixed at origin and the distances along the bar be denoted by x, then the equation of heat flow (in one dimension) may be expressed as

$$\frac{\partial^2 u}{dx^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad \text{... (1)}$$

The boundary conditions are

$$\left.\begin{array}{l} u = 0 \text{ when } x = 0 \\ u = 0 \text{ when } x = l \end{array}\right\} \text{for all values of } t \qquad \text{... (2)}$$

$$u = f(x) \text{ at } t = 0 \text{ and } u \neq \infty \text{ for } t = \infty \qquad \text{... (3)}$$

Let the solution of equation (1) be expressed as

$$u(x, t) = \phi(x)\,\tau(t) \qquad \text{... (4)}$$

Substituting this in (1) and dividing throughout by $\phi\,\tau$; we get

$$\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{h^2\,\tau}\frac{\partial \tau}{\partial t} \qquad \text{... (5)}$$

In this equation left hand side is the function of x only while right hand side is the function of t only ; therefore if above equation is satisfied, each side must be equal to a constant $-\alpha^2$ (say), *i.e.*

$$\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2} = -\alpha^2 \text{ or } \frac{\partial^2 \phi}{\partial x^2} + \alpha^2 \phi = 0 \qquad \text{... (6)}$$

and
$$\frac{1}{h^2 \tau}\frac{\partial \tau}{\partial t} = -\alpha^2 \text{ or } \frac{\delta \tau}{\tau} = -\alpha^2 h^2\,\delta t \qquad \text{... (7)}$$

Integrating (7) ; we obtain

$$\log \tau = -\alpha^2 h^2\,t + \log C \qquad \text{(log } C \text{ being constant of integration)}$$

This gives

$$\tau = C\,e^{-\alpha^2 h^2\,t} \qquad \text{... (8)}$$

The general solution of equation (6) is

$$\phi = A \sin \alpha x + B \cos \alpha x \quad \ldots (9)$$

Now ϕ must satisfy boundary conditions (2). The first condition $\phi = 0$ at $x = 0$ gives $B = 0$. In order that $\phi = 0$ at $x = l$; we must have

$A \sin \alpha l = 0$ or $\alpha l = r\pi$ for a non-trivial solution $(r = 0, 1, 2, 3 ...)$

This gives the allowed values of α as

$$\alpha = \frac{r\pi}{l} \quad \ldots (10)$$

For each value of r, there corresponds a solution of differential equation (6) of the form

$$\phi_r = A_r \sin \frac{r\pi x}{l} \quad \ldots (11)$$

where A_r is an arbitrary constant.

Substituting value of α from (10) in (8) ; we get

$$\tau = C\, e^{-(r\pi h/l)^2 t} \quad \ldots (12)$$

Thus we see from (4), (11) and (12) that the solution of equation (1) for each value of r is of the form

$$u_r = A_r\, C\, e^{-(r\pi h/l)^2 t} \sin \frac{r\pi x}{l} = N_r\, e^{-(r\pi h/l)^2 t} \sin \frac{r\pi x}{l} \quad \ldots (13)$$

where $N_r = A_r\, C$ new aribitrary constant.

By summing over for all values of r, the general solution of equation (1) may be expressed as

$$u = \sum_{r=1}^{\infty} N_r\, e^{-(r\pi h/l)^2 t} \sin \frac{r\pi x}{l} \quad \ldots (14)$$

(Summation starts from $r = 1$ since term corresponding to $r = 0$ vanishes)

The constant N_r is determined using initial condition (3) *i.e.* $u = f(x)$ at $t = 0$. Using this equation (14) gives

$$f(x) = \sum_{r=1}^{\infty} N_r \sin \frac{r\pi x}{l} \quad \ldots (15)$$

which is *Fourier sine series.* To evaluate A_r we multiply both sides by $\sin \frac{m\pi x}{l}$ and integrate between limits 0 to l and obtain,

$$\int_0^l f(x) \sin \frac{m\pi x}{l}\, dx = \sum_{r=1}^{\infty} N_r \int_0^l \sin \frac{r\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \sum_{r=1}^{\infty} N_r \frac{l}{2} \delta_{m,r} = N_m \cdot \frac{l}{2}$$

i.e. $$N_m = \frac{2}{l} \int_0^l f(x) \sin \frac{m\pi x}{l}\, dx$$

Replacing m by r, we obtain

$$N_r = \frac{2}{l} \int_0^l f(x) \sin \frac{r\pi x}{l}\, dx \quad \ldots (16)$$

Substituting this value of N_r in equation (14) we get the desired solution of differential equation (1).

If *instead of the ends of the bar being kept at temperature zero, they are impervious to heat* : then the boundary conditions of the problem become

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 0 \text{ at } x = 0 \\ \frac{\partial u}{\partial x} &= 0 \text{ at } x = l \end{aligned}\right\} \text{for all values of } t \qquad \text{... (17)}$$

$$u = f(x) \text{ for } t = 0,\ u \neq \infty \text{ at } t = \infty \qquad \text{... (18)}$$

In this case we have $u = C\, e^{-\alpha^2 h^2 t}\, \phi(x)$; where

$$\phi = A \sin \alpha x + B \cos \alpha x$$

so

$$\frac{\partial \phi}{\partial x} = \alpha A \cos \alpha x - \alpha B \sin \alpha x \qquad \text{...(19)}$$

Now, first boundary condition $\frac{\partial u}{\partial x} = 0$ at $x = 0$, gives $A = 0$

and the second boundary condition $\frac{\partial u}{\partial x}$ at $x = 0$ gives

$$B \sin \alpha l = 0$$

For a non-trivial solution $B \neq 0$; therefore $\sin \alpha l = 0$

or $\alpha l = r\pi$ *i.e* $\alpha = \frac{r\pi}{l}$, $r = 0, 1, 2, 3, \ldots\ldots$ ...(20)

Continuing the same reasoning as before, we obtain the general solution

$$u = N_0 + \sum_{r=1}^{\infty} N_r\, e^{-(r\pi h/l)^2 t} \cos \frac{r\pi x}{l} \qquad \text{... (21)}$$

At $t = 0$, $u = f(x)$ gives

$$f(x) = N_0 + \sum_{r=1}^{\infty} N_r \cos \frac{r\pi x}{l} \qquad \text{... (22)}$$

This is half *range cosine series.* The constant N_0 is determined by integrating (22) between limits $(0, l)$

$$i.e. \quad \int_0^l f(x)\, dx = N_0 \int_0^l dx \ \ i.e.\ N_0 = \frac{1}{l} \int_0^l f(x)\, dx \qquad \text{... (23)}$$

The constant N_r is determined by multiplying both sides of (22) by $\cos \frac{r\pi x}{l}$ and integrating between limits $(0, l)$. This gives

$$N_r = \frac{2}{l} \int_0^l f(x) \cos \frac{r\pi x}{l}\, dx \qquad \text{... (24)}$$

Substituting values of N_0 and N_r in (21) ; we get the desired solution.

Ex. 12. *The ends A and B of a rod 20 cm long are at temperatures 30° C and 80° C respectively until steady state prevails. The temperatures at the ends are changed to 40° C and 60° C respectively. Find the temperature distribution in the rod at time t.* *(Delhi 1999)*

Solution. In steady state the temperature gradient remains same throughout the rod. Initial

$$\text{Temperature gradient} = \frac{\theta_1 - \theta_2}{l} = \frac{80 - 30}{20} = 2{\cdot}5\ ^\circ\text{C/cm}$$

$\therefore$ Initial temperature distribution in rod (in steady state)

$$u = 30 + 2{\cdot}5\, x \qquad \text{... (1)}$$

when $n = 0$

$$A_0 = 100 \times \left(\frac{1}{2}\right) \int_0^1 P_0(x)\, dx = 50 \int_0^1 1.dx = 50$$

when $n = 1$

$$A_1 = 100 \times \left(\frac{3}{2}\right) \int_0^1 P_1(x)\, dx = 150 \int_0^1 x\, dx = 150 \times \left[\frac{x^2}{2}\right]_0^1 = 75$$

when $n = 2$

$$A_2 = 100 \times \left(\frac{5}{2}\right) \int_0^2 P_2(x)\, dx = 250 \int_0^2 \frac{1}{2}(3x^2 - 1)\, dx = \frac{250}{2}\left[\frac{3x^3}{3} - x\right]_0^1 = 0$$

when $n = 3$

$$A_3 = 100 \times \frac{7}{2} \int_0^1 P_3(x)\, dx = 350 \int_0^1 \left(\frac{5x^3 - 3x}{2}\right) dx = \frac{350}{2}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1$$

$$= \frac{350}{2}\left[\frac{5}{4} - \frac{3}{2}\right] = -\frac{175}{4}$$

and so on.

$$\therefore \quad u = \sum_{n=1}^{\infty} A_n P_n(\cos\theta) = \left[50 + 75 P_1(\cos\theta) - \frac{175}{4} P_3(\cos\theta) + \ldots\right]$$

This is required temperature distribution.

9·14. The Equation of Motion for the Vibrating String

A string is a cord or wire whose length is very large as compared to its diameter and which is perfectly uniform and flexible. When a string is stretched between two points with a large tension, plucked transverse vibrations are produced in it. In order to simplify the problem, let us assume that the string vibrates only in vertical plane.

Consider the motion of an element PQ of the string of length dl. Let O be the origin and (x, y) and $(x + dx, y + dy)$, the coordinates of points P and Q respectively. Let $T(x)$ and $T(x + dx)$ be the tensions at P and Q respectively and $\theta(x)$ and $\theta(x + dx)$ the angles which the tangents at P and Q make with X-axis. If $\hat{i}$ and $\hat{j}$ are unit vectors along X and Y-axes respectively, then the net horizontal force along X-axis acting on the element PQ

$$= [T(x + dx)\cos\theta(x + dx) - T(x)\cos\theta(x)]\,\hat{i} \quad \ldots (1)$$

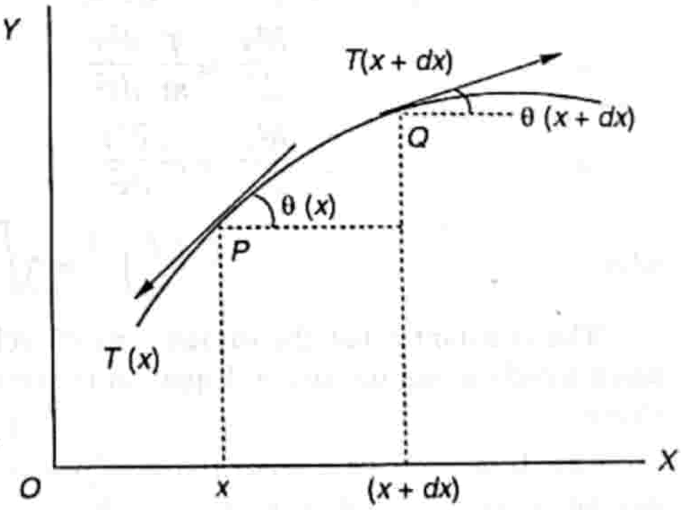

Fig. 9·6

The net vertical force along Y-axis acting on the element PQ

$$= [T(x + dx)\sin\theta(x + dx) - T(x)\sin\theta(x)]\,\hat{j} \quad \ldots (2)$$

Assuming that the vibrations take place in the vertical plane only, the horizontal motion is negligible and hence, force represented by (1) is zero.

If m is the mass per unit length of the string, then mass of the element $PQ = m\,dl$.

According to Newton's law, the equation of motion, neglecting all other forces, is given by

$$m\,dl\,\frac{\partial^2 y}{dt^2}\;\hat{\mathbf{j}} = [T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)]\;\hat{\mathbf{j}}.$$

Dividing by $dx\;\hat{\mathbf{j}}$ we get

$$m\,\frac{dl}{dx}\cdot\frac{\partial^2 y}{dt^2} = \frac{T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)}{dx}$$

But $\qquad dl^2 = dx^2 + dy^2$

$$\therefore \quad m\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}\frac{\partial^2 y}{dt^2} = \frac{T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)}{dx}$$

Taking the limit $dx \to 0$; the above equation can be written as

$$m\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}\frac{\partial^2 y}{dt^2} = \frac{\partial}{dx}(T\sin\theta). \qquad \ldots (3)$$

But $\qquad \sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\partial y/\partial x}{\sqrt{1+(\partial y/\partial x)^2}}.$

Then equation (3) can be written as

$$m\,\frac{\partial^2 y}{dt^2}\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} = \frac{\partial}{\partial x}\left[T\,\frac{\partial y/\partial x}{\sqrt{1+(\partial y/\partial x)^2}}\right].$$

Let us restrict ourselves to *small vibrations,* so that the slope $\dfrac{\partial y}{\partial x}$ is small compared to unity, then we have

$$m\,\frac{\partial^2 y}{dt^2} = \frac{\partial}{\partial x}\left[T\left(\frac{\partial y}{\partial x}\right)\right] \qquad \ldots (4)$$

Let us further assume that the *tension T is constant throughout the string,* then

$$m\,\frac{\partial^2 y}{\partial t^2} = T\,\frac{\partial^2 y}{dx^2}$$

or

$$\frac{\partial^2 y}{dt^2} = \frac{T}{m}\,\frac{\partial^2 y}{dx^2} \qquad \ldots (5)$$

or

$$\frac{\partial^2 y}{dt^2} = c^2\,\frac{\partial^2 y}{dx^2} \qquad \ldots (6)$$

where

$$c = \left(\frac{T}{m}\right)^{1/2} = \sqrt{\frac{T}{m}} \qquad \ldots (7)$$

The constant c has the dimensions of velocity and actually it is the velocity which the wave travels along the string. Equation (6) represents the equation of motion of the vibtrating string.

Cor. If in *addition a vertical force f per unit length* acts on the string, then the equation of motion of the vibrating string is given by

$$m\,dl\,\frac{\partial^2 y}{dt^2}\;\hat{\mathbf{j}} = [T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)]\;\hat{\mathbf{j}} + f\cdot dl\;\hat{\mathbf{j}}$$

which yields as before

$$m\,\frac{\partial^2 y}{dt^2} = T\,\frac{\partial^2 y}{dx^2} + f \qquad \ldots (8)$$

If f is the gravitational force per unit length of string, then $f = -mg$ and the equation of motion for the vibrating string will be

$$\frac{\partial^2 y}{dt^2} = c^2 \frac{\partial^2 y}{dx^2} - g \qquad \text{... (8)}$$

9·15. D' Alembert's Solution

The wave equation for the vibrating string is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad \text{... (1)}$$

Obviously, this displacement y is the function of x and t, *i.e.*

$$y = y(x, t) \qquad \text{... (2)}$$

Let us introduce the two new variables u and v such that

$$u = x + ct \text{ and } v = x - ct \qquad \text{... (3)}$$

Evidently,

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial x} = 1 \\ \text{and} \quad \frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial t} = c \end{aligned}\right\} \qquad \text{... (4)}$$

From (2) and (3), it is obvious that y may be considered as the function of new variables u and v *i.e.*

$$y = y(u, v) \qquad \text{... (5)}$$

This implies

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} \cdot 1 + \frac{\partial y}{\partial v} \cdot 1 = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) y \qquad \text{... (6)}$$

From which

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{... (7)}$$

Similarly,

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial t} = \frac{\partial y}{\partial u} \cdot c + \left(\frac{\partial y}{\partial v}\right) \cdot (-c) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) y \qquad \text{... (8)}$$

From which

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \qquad \text{... (9)}$$

Also

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} + 2\frac{\partial^2 y}{\partial u\, \partial v}$$

and

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) c\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= c^2\left\{\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} - 2\frac{\partial^2 y}{\partial u\, \partial v}\right\} \qquad \text{... (11)}$$

Using (10) and (11) ; equation (1) gives

$$\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} + 2\frac{\partial^2 y}{\partial u\, \partial v} = \frac{1}{c^2} \cdot c^2 \left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} - 2\frac{\partial^2 y}{\partial u\, \partial v}\right) \qquad \text{... (12)}$$

or

$$4 \cdot \frac{\partial^2 y}{\partial u\, \partial v} = 0 \text{ i.e. } \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0$$

Integrating this with respect to u ; we obtain

$\frac{\partial y}{\partial v} = f(v)$; the constant of integration.

Integrating again with respect to v (now) ; we get

$$y = F_1(u) + \int f(v)\, dv = F_1(u) + F_2(v)$$

where F_1 and F_2 are arbitrary functions of u and v respectively. Substituting values of u and v from (3), we get

$$y = F_1(x + ct) + F_2(x - ct) \quad \text{... (13)}$$

This is called *D' Alembert's solution of vibrating string.*

Physical Interpretation. If $F_1(x + ct)$ is plotted against x, the curve has exactly the same form as that of $F_1(x)$ but every point on it is displaced a distance ct to the left of the corresponding point in $F_1(x)$. The function $F_1(x + ct)$ thus represents a wave of displacement of arbitrary shape travelling towards the left along the string with the same speed c. In the same manner $F_2(x - ct)$ represents a wave of displacement travelling to the right along the string with the same speed c. Thus ***the general solution represents the sum of these two waves travelling in opposite directions with the same speed c.***

Cor. Consider a string of length l both ends fixed. Suppose a wave of arbitrary shape given by $y = F(ct + x)$ ... (14)

is approaching the origin ($x = 0$). At the origin the displacement must be of the form

$y = -F(ct - x)$ ... (15)

Since the sum of (14) plus (15) is zero at $x = 0$ for all t. This shows that the *transverse waves in stretched string are inverted by reflection from a fixed end.*

9.16. Fourier Series Solution *(The Method of Separation of Variables)*

To find the solution of wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ... (1)

we require certain initial conditions and restrictions of the string, known as *boundary conditions.*

If we take initial instant $t = 0$, then boundary conditions may be written as

$$\left.\begin{aligned} &\text{initial position } y_0(x) = y(x, 0) \\ \text{and} \quad &\text{initial velocity } v_0(x) = \left[\frac{\partial y}{\partial t}\right]_{t=0} \end{aligned}\right\} \quad \text{.... (2)}$$

If the string is fixed at its ends, then we also have

$$y(0, t) = y(l, t) = 0. \quad \text{... (3)}$$

To find the solution of (1) by the method of separation of variables, let us consider that the solution is of the form

$$y(x, t) = X(x)\, \phi(t), \quad \text{... (4)}$$

where X is the function of x only and ϕ the function of t only. Then we have

$$\frac{\partial y}{\partial x} = \phi \frac{\partial X}{\partial x}, \quad \frac{\partial^2 y}{\partial x^2} = \phi \frac{\partial^2 X}{\partial x^2}$$

and

$$\frac{\partial y}{\partial t} = X \frac{\partial \phi}{\partial t}, \quad \frac{\partial^2 y}{\partial t^2} = X \frac{\partial^2 \phi}{\partial t^2}$$