Linearly dependent and linearly independent sets:

Linearly dependent vectors:

Let V be a vector space over a field F. Then the vectors $v_1, v_2, ---, v_n$ of V are said to be linearly dependent if there exist scalars $a_1, a_2, ---, a_n \in F$ such that

$$a_1 v_1 + a_2 v_2 + --- + a_n v_n = 0$$

Linearly independent vectors

Let V be a vector space over a field F. Then the vectors $v_1, v_2, ---, v_n$ are said to be linearly independent (LI) if $a_1, a_2, ---, a_n \in F$ such that

$$a_1 v_1 + a_2 v_2 + --- + a_n v_n = 0$$

$$\Rightarrow a_1 = 0, a_2 = 0, ----, a_n = 0.$$

<u>NOTE</u>: Let V be a vector space a field F and let $S = \{v_1, v_2, --, v_n\}$ be a subset of V. Then V is linearly dependent (linearly independent) if all the members of S are linearly dependent (linearly independent}

Example

(1) Consider $V = \mathbb{R}^2$ and let

$v_1 = (1, 2)$, $v_2 = (2, 0)$, $v_3 = (8, 4)$

Clearly, v_1, v_2 and v_3 are linearly dependent

as $\quad 2v_1 + 3v_2 - v_3 = 0$

(2) $\quad v_1 = (1, 2)$, $v_2 = (4, 8)$

Clearly, v_1 and v_2 are LD as

$4v_1 - v_2 = 0$

(3) Consider $V = \mathbb{R}^3$.

$v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$

Then v_1, v_2 and v_3 are linearly independent.

as $\quad a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

i.e $\quad a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = 0$

$\Rightarrow \quad a_1 = 0, \; a_2 = 0, \; a_3 = 0.$

(4) $\quad V = \mathbb{R}^3$. $\quad v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$

$v_3 = (1, 0, 0)$

Let x, y, z be scalars s.t.

$x v_1 + y v_2 + z v_3 = 0$

$x(1, 1, 1) + y(1, 1, 0) + z(1, 0, 0) = 0$

$$\Rightarrow \quad \begin{aligned} x+y+z &= 0 \\ x+y &= 0 \\ x &= 0 \end{aligned}$$

Solving above equations, we get
$x=0,\ y=0,\ z=0$.

Hence the vectors are LI.

Que: Determine whether or not the set $\{(2,-1,0),(1,2,5),(7,-1,5)\}$ is linearly independent over $\mathbb{R}$.

Solution Let x, y, z be real number s.t.

$$x(2,-1,0)+y(1,2,5)+z(7,-1,5)=0$$

$$\therefore \quad \begin{aligned} 2x+y+7z &= 0 \quad \text{——} (1) \\ -x+2y-z &= 0 \quad \text{——} (2) \\ 5y+5z &= 0 \quad \text{——} (3) \end{aligned}$$

Now we will ~~to~~ solve these equations for x, y, z.

Coefficient matrix, $A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \end{bmatrix}$

$$\det(A) = 2(10+5) - 1(-5-0) + 7(-5-0)$$

$$= 30+5-35 = 0.$$

$\therefore$ The equations (1), (2), (3) has a non-zero solution and hence the vectors are LD.

Remarks :

1) Every singleton set, having a non-zero vector is linearly independent.
2) Two vectors are linearly dependent iff one of them is a multiple of the other.
3) A set containing zero vector is always linearly dependent.

Que: Determine which sets are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

i) $S_1 = \{(1,0,0), (1,1,0), (1,1,1)\}$
ii) $S_2 = \{(1,0,1), (0,0,0), (0,1,0)\}$
iii) $S_3 = \{(1,0,-2), (3,2,-4), (-3,-5,1)\}$
iv) $S_4 = \{(2,-2,1), (1,-3,2), (-7,5,4)\}$
v) $S_5 = \{(1,-3,0), (-2,9,0), (0,0,0), (0,-3,5)\}$
vi) $S_6 = \{(1,2,-3), (-4,-5,6)\}$
vii) $S_7 = \{(-2,3,0), (-6,-1,5)\}$