Dimension of Subspaces:-

Let V be a vector space and W be a ~~finite~~ subspace of V such that W has basis consisting of finite no. of vectors, then the unique no. of vectors in each basis for W is called dimension of W.

Example:-

(1) Let W be the set of $n \times n$ diagonal matrices then W is a subspace of $M_{n\times n}(\mathbb{R})$

and $\mathcal{B} = \{E^{11}, E^{22}, \ldots, E^{nn}\}$

is a basis for W, where E^{ij} is the matrix in which only non zero entry is a 1 in i^{th} row and j^{th} column.

$\therefore \dim(W) = n$.

Note that $\dim(M_{n\times n}(\mathbb{R}))$ is n^2.

(2) Let $W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_2 = a_3 = a_4 \ \& \ a_1 + a_5 = 0\}$

then W is a subspace of $\mathbb{R}^5$ (show) and $B = \{(1,0,0,0,-1), (0,1,1,1,0)\}$

is a basis of W. (show it)

$\therefore$ $\dim(W) = 2$.

Note that $\dim(\mathbb{R}^5) = 5$.

(3.) Let W be the set of 3×3 symmetric matrices then W is a subspace of $M_{3\times3}(\mathbb{R})$.

and
$$B = \left\{ \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A^{11}}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A^{12}}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A^{13}, \right.$$
$$\left. A^{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A^{23}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A^{33}} \right\}.$$

is a basis for W.

$\therefore$ $\dim(W) = 6$

Whereas $\dim(M_{3\times3}(\mathbb{R})) = 9$.

(4.) Let W be set of $n\times n$ symmetric matrices. then W is a subspace of $M_{n\times n}(\mathbb{R})$

and

$B = \{A^{ij} : 1 \le i \le j \le n\}$

where A^{ij} is $n\times n$ matrix, having 1 in i^{th} row & j^{th} column and 1 in j^{th} row & i^{th} column and 0 elsewhere.

~~so~~ then $\mathcal{B}$ is a basis for W.

and $\mathcal{B} = \{ A^{11}, A^{12}, \ldots, A^{1n}$

$A^{22}, A^{23}, \ldots, A^{2n}$

$A^{33}, \ldots A^{3n}$

$\vdots$

$A^{nn} \}$

(As $A^{ij} = A^{ji}$
A is symmetric).

$\therefore \dim(W) = n + (n-1) + \ldots + 1$

$$= \frac{n(n-1)}{2}.$$

Also, $\dim(M_{n\times n}(\mathbb{R})) = n^2$.

Observing all these examples, we have if W be a subspace of finite dimensional vector space V, then $\dim(W) \le \dim V$.

<u>Thm 1.11</u> :- Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \le \dim(V)$. Moreover, if $\dim(W) = \dim(V)$ then $V = W$.

<u>Proof</u>:- Let $\dim(V) = n$.

If $W = \{0\}$, then W is finite dimensional and $\dim(W) = 0 \le n$.

Else W contains a non-zero vector x_1

and $\{x_1\}$ is a linearly independent set,
Continue choosing $x_1, x_2, \ldots, x_k$ in W such that $\{x_1, x_2, \ldots, x_k\}$ is linearly independent and is largest linearly independent set, that is adding any other vector from W produces a linearly dependent set.

And we must find such k, as no linearly independent subset of V (as $W \subseteq V$) contains ~~...~~ more than n vectors.

$$\therefore \quad k \leq n.$$

Thus By thm (1.7),

$\{x_1, x_2, \ldots x_k\}$ generates W

$\therefore$ $\{x_1, x_2, \ldots, x_k\}$ is a basis for W.

$\therefore$ $\dim(W) = k \leq n$.

And if $\dim(W) = n = \dim(V)$
then a basis for W contains n linearly independent vectors
$\therefore$ By replacement theorem (Corollary 2)
that basis for W is also a basis for V.

$\therefore \quad W = V$ ■

Corollary:- If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

Proof:- Let β be a basis for W,
then β is a linearly independent subset of W, therefore subset of V.
Then By Replacement theorem (Corollary 2),
β can be extended to a basis for V. ■.

Question 1.!- Determine the subspaces of $\mathbb{R}^2$.

→ Let W be a subspace of $\mathbb{R}^2$
then $\dim(W)$ is either 0, 1 or 2.

If $\dim(W) = 0$, then $W = \{0\}$

If $\dim(W) = 2$, then $W = \mathbb{R}^2$.

and If $\dim(W) = 1$.
Let $x \in W$ be a non-zero vector.
then $\{x\}$ is linearly independent set of W.
$\therefore \{x\}$ is a basis for W.
$\therefore \quad W = \{\alpha x : \alpha \in \mathbb{R}\}$

$\therefore$ W is all scalar multiples of x.

(Geometrically W is a line passing through point 0 and x).

$\therefore$ Any subspace of $\mathbb{R}^2$ having dimension 1 consists of all scalar multiples of some non-zero vector in $\mathbb{R}^2$.

Q^n Determine all subspace of $\mathbb{R}^3$. (Do it).

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Q12 Let u, v and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Sol^n Let $\{u, v, w\}$ be a basis for V.

To show $\{u+v+w, v+w, w\}$ is a basis for V, it is sufficient to show that this subset is linearly independent. (Why is that)

Let a_1, a_2 and a_3 be scalars such that

$$a_1(u+v+w) + a_2(v+w) + a_3 w = 0.$$

$$\Rightarrow a_1 u + (a_1+a_2)v + (a_1+a_2+a_3)w = 0$$

$\therefore$ $a_1 = 0$

$a_1 + a_2 = 0$

$a_1 + a_2 + a_3 = 0$

[As $\{u, v, w\}$ is linearly independent]

$\therefore$ we get $a_1 = 0$, $a_2 = 0$ & $a_3 = 0$

Thus $\{u+v+w, v+w, w\}$ is linearly independent subset of V.

$\Rightarrow$ $\{u+v+w, v+w, w\}$ is a basis for V. ■

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Q15 Let W be the set of all $n \times n$ matrices having trace equal to zero.

then W is a subspace of $M_{n \times n}(F)$.

Find basis for W and $\dim(W)$.

Sol^n:- We will do it for general case, first see if $n = 3$ what will happen.

Let W = set of 3×3 matrices having trace 0.

then $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right.$

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$

is a basis for W.

and $\dim(W) = 8$.

Now we do for general $n \times n$ matrices.

then

$$B = \{ E^{ij}, A^{11}, A^{22}, \ldots A^{(n-1)(n-1)} : 1 \leq i, j \leq n \ \& \ i \neq j \}$$

is a basis for W, where

E^{ij} is $n \times n$ matrix having 1 at i^{th} row and j^{th} column and every other is 0.

and A^{ii} is $n \times n$ diagonal matrix having 1 at i^{th} row & i^{th} column and -1 at n^{th} row n^{th} column $(\forall \ 1 \leq i \leq n-1)$.

$\therefore \ \dim(W) = n^2 - 1$.