Differential Equations

Recommended Books:

(1) Shepley L. Ross - Differential equations, Third edition John Willey & Sons, 1984.

(2) Daniel A. Murray - Introductory Course in Differential equations

(3) I. Sneddon - Elements of Partial Differential equations, ~~McM~~ McGraw Hill, International Edition, 1967.

Ordinary Differential Equations (ODEs)

First Order Differential Equations :—

A differential equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{——— (1)}$$

or

$$M(x, y)\,dx + N(x, y)\,dy = 0 \quad \text{—(2)}$$

is called first order DEs.

Ex:

$$\frac{dy}{dx} = \frac{x^2+y^2}{x-y} \quad \text{or} \quad (x^2+y^2)\,dx + (x-y)\,dy = 0.$$

Eqn (1) is called derivative form and eqn (2) is called differential form.

Exact Differential Equations :-

Definition: A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$ is

Called exact differential equation if there exists a function f such that

$$\boxed{df(x,y) = M(x,y)\,dx + N(x,y)\,dy} \quad \text{--- (2)}$$

And we know that

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy \quad \text{--- (3)}$$

From (2) & (3), we have

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \quad \& \quad \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

and

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

Hence

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

Therefore A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ is exact

iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$ for all $(x,y) \in D$

Solution of exact differential equations :-

If the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$

is exact, then sol^n of (1) is given as

$$\underset{[\text{Treat } y \text{ as constant}]}{\int M(x,y)\,dx} + \int [\text{Only take those term of } N(x,y) \text{ which doesnot have } x]\,dy = C$$

Example; (1) $(3x^2+4xy)dx + (2x^2+2y)dy = 0$ --- (1)

$M(x,y) = 3x^2+4xy$, $N(x,y) = 2x^2+2y$

$\frac{\partial M}{\partial y} = 4x$, $\frac{\partial N}{\partial x} = 4x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ eq^n (1) is exact.

Hence sol^n of (1) is

$$\underset{[\text{Treat } y \text{ as Constant}]}{\int (3x^2+4xy)\,dx} + \int 2y\,dy = C$$

$$3\frac{x^3}{3} + 4\frac{x^2}{2}\cdot y + 2\frac{y^2}{2} = C$$

$$\boxed{x^3 + 2x^2y + y^2 = C}$$

Example (2) Solve the initial-Value problem

$$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0,$$
$$y(0) = 2$$

$$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0 \quad \text{--- (1)}$$

$M = (2x\cos y + 3x^2 y)$, $N = x^3 - x^2\sin y - y$

$$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2, \quad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\Rightarrow$ eqn (1) is exact.

Hence soln of (1) is given as

$$\int (2x\cos y + 3x^2 y)\,dx + \int [(-y)]\,dy = c$$

[Treat y as constant]

$$2\frac{x^2}{2}\cos y + 3\frac{x^3}{3}y - \frac{y^2}{2} = c$$

$$\boxed{x^2\cos y + x^3 y - \frac{y^2}{2} = c} \quad \text{--- (2)}$$

given that $y(0) = 2$, Putting this in eqn (2), we got

$$0\times\cos(2) + 0\times 2 - \frac{2^2}{2} = c$$

$$\boxed{c = -2}$$

Hence soln of given Initial value Problem (IVP) is

$$\boxed{x^2\cos y + x^3 y - \frac{y^2}{2} = -2}$$

Ans

Exercise (Exact Differential Equations).

(1) $(3x+2y)\,dx+(2x+y)\,dy=0$

(2) $(y^2+3)\,dx+(2xy-4)\,dy=0$

(3) $(2xy+1)\,dx+(x^2+4y)\,dy=0$

(4) $(3x^2y+2)\,dx - (x^3+y)\,dy=0$

(5) $(6xy+2y^2-5)\,dx+(3x^2+4xy-6)\,dy=0$

(6) $(\theta^2+1)\cos x\,dx+2\theta\sin x\,d\theta=0$

(7) $(y\sec^2 x+\sec x\tan x)\,dx+(\tan x+2y)\,dy=0$

(8) $\left(\frac{x}{y^2}+x\right)dx+\left(\frac{x^2}{y^3}+y\right)dy=0$

(9) $\left(\frac{2s-1}{t}\right)ds+\left(\frac{s-s^2}{t^2}\right)dt=0$

(10) $\left(\frac{2y^{3/2}+1}{x^{1/2}}\right)dx+\left(3x^{1/2}y^{1/2}-1\right)dy=0$.

Solve the following IVP (Initial value Problems)

(11) $(2xy-3)\,dx+(x^2+4y)\,dy=0,\quad y(1)=2.$

(12) $(3x^2y^2-y^3+2x)\,dx+(2x^3y-3xy^2+1)\,dy,\quad y(-2)=1$

(13) $(2y\sin x\cos x+y^2\sin x)\,dx+(\sin^2 x-2y\cos x)\,dy$

(14) $(ye^x+2e^x+y^2)\,dx+(e^x+2xy)\,dy=0,\quad y(0)=3.$

(15) $\left(\frac{3-y}{x^2}\right)dx+\left(\frac{y^2-2x}{xy^2}\right)dy=0,\quad y(0)=6.$ $y(-1)=2.$

(16) $\left(\frac{1+8xy^{2/3}}{x^{2/3}y^{1/3}}\right)dx+\left(\frac{2x^{4/3}y^{2/3}-x^{1/3}}{y^{4/3}}\right)dy=0,\quad y(1)=8.$