B.Sc. APS with C.S.
Semester – VI

Markov's Inequality

Statement: If X is a nonnegative random variable, then for any $a > 0$, $P[X \geq a] \leq \frac{E(X)}{a}$

Proof:

We shall prove this result for continuous random variable X. Similarity it can be proved for discrete random variable X.

Let X be a continuous random variable having density function $f(x)$ and given that X is a nonnegative random variable. Therefore, X can take only nonnegative values.

Now $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{0}^{\infty} x f(x)\, dx \qquad [\because X \text{ is nonnegative}]$$

$$= \int_{0}^{a} x f(x)\, dx + \int_{a}^{\infty} x f(x)\, dx \qquad [\because a > 0]$$

$$\geq \int_{a}^{\infty} x f(x)\, dx$$

$$\geq \int_{a}^{\infty} a\, f(x)\, dx \qquad \left[\because \text{for limits from } a \text{ to } \infty \quad x \geq a\right]$$

$\therefore E(X) \geq a \int_a^\infty f(x)\,dx$

$\Rightarrow E(X) \geq a P(X \geq a) \Rightarrow a P(X \geq a) \leq E(X)$

$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$

Chebyshev's Inequality

Statement:

Let X be a random variable with mean μ and variance σ^2. then for any $k > 0$.

$$P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2} \quad \text{——— (1)}$$

Proof:

We know that $(X-\mu)^2$ is a non-negative random variable.

So replacing X by $(X-\mu)^2$ and a by k^2 in Markov's inequality, we have

$$P[(X-\mu)^2 \geq k^2] \leq \frac{E[(X-\mu)^2]}{k^2}$$

$$\Rightarrow P[|X-\mu| \geq k] \leq \frac{V(X)}{k^2} \quad \left[\begin{array}{l} \because x^2 \geq a^2 \\ \Rightarrow |x| \geq a \text{ for any } a > 0 \\ \& \; E(X-\mu)^2 = V(X) \end{array}\right]$$

$$\Rightarrow P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2} \quad [\because V(X) = \sigma^2]$$

Remarks:

① $P[|X-\mu| < k] \geq 1 - \frac{\sigma^2}{k^2}$

Proof:

$$P[|X-\mu| \leq k] = 1 - P[|X-\mu| \geq k] \geq 1 - \frac{\sigma^2}{k^2}$$

② By replacing k by $k\sigma$ in eqn ①, Chebyshev's inequality can be expressed as

$$P[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

③ $P[|X-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$

④ The Chebyshev's inequality gives a bound for the probability of a event given in terms of a random variable and its mean and variance. This bound is independent of distribution of X. Therefore, the use of this inequality is recommended in situations where the form of the distribution is not known.

⑤ Chebyshev's inequality holds for any distribution of observations and for this reason, the results are usually weak.

Example: Let X has the pdf

$$f(x) = \begin{cases} 630x^4(1-x)^4, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

find the probability that it will take on a value within two standard deviations of mean and compare this probability with the lower bound provided by Chebyshev's inequality.

Soln

Given $f(x) = \begin{cases} 630x^4(1-x)^4, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

mean, $\mu = E(X) = \int_0^1 x f(x)\,dx = 1/2 = 0.5$

$E(X^2) = \int_0^1 x^2 f(x)\,dx = 3/11$

Hence, variance is $V(X) = \sigma^2 = E(X^2) - (E(X))^2 = \frac{3}{11} - \frac{1}{4}$

$\Rightarrow \sigma^2 = \frac{1}{44}$

$\Rightarrow \sigma = \sqrt{\frac{1}{44}} \cong 0.15$

Now, the probability that X will take on a value within two standard deviations of the mean is given by

Pr: $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0.5 - 2(0.15) < X < 0.5 + 2(0.15))$

$= P(0.20 < X < 0.80)$

$= \int_{0.2}^{0.8} 630x^4(1-x)^4\,dx = 0.96$ ——— (1)

Now using Chebyshev's inequality, we have

$$P(\mu-2\sigma < X < \mu+2\sigma) \cancel{= P(-2\sigma < X < 2\sigma)}$$

$$= P(-2\sigma < X-\mu < 2\sigma) = P(|X-\mu| < 2\sigma)$$

$$\geq 1 - \frac{1}{(2)^2} \quad \left[\text{from remarks no. 3}\right]$$

$$= 1 - \frac{1}{4} = 0.75$$

$$\therefore P(|X-\mu| < 2\sigma) = P(0.2 < X < 0.8) \geq 0.75 \quad \text{—— (2)}$$

We can easily see that the statement (1) is much stronger than statement (2), as we have already mentioned in remarks no. 5 that the results from Chebyshev's inequality are usually weak. But it is very useful when distribution is not known.

Example:-

Suppose the number of items produced in a factory during a week is a random variable with mean 500. Then

(i) What can be said about the probability that this week's production will be at least 1000?

(ii) If the variance of the week's production is known to 100, then what can be said about the probability that this week's production will be between 400 and 600?

Solⁿ Let X be the no. of items produced in a factory during a week.

∴ Given Mean $= E(X) = \mu = 500$

(i) $P(X \geq 1000) \leq \frac{E(X)}{1000}$ [Using Markov's inequality]

$= \frac{500}{1000} = \frac{1}{2}$

$\therefore P(X \geq 1000) \leq 0.5$

(ii) Given that $\sigma^2 = 100$ & $\mu = 500$

Now $P(400 \leq X \leq 600) = P(400 - 500 \leq X - 500 \leq 600 - 500)$

$= P(-100 \leq X - 500 \leq 100)$

$= P(|X - 500| \leq 100)$ —— (1)

Now $P(|X-500| \geq 100) \leq \frac{\sigma^2}{(100)^2} = \frac{100}{(100)^2} = \frac{1}{100}$ —— (2)

using Chebyshev's inequality.

Now from eqⁿ (1)

$\therefore P(400 \leq X \leq 600) = P(|X-500| \leq 100)$

$= 1 - P(|X-500| \geq 100)$

$\geq 1 - \frac{1}{100}$ [using eqⁿ (2)]

$= \frac{99}{100} = 0.99$

$\therefore P(400 \leq X \leq 600) \geq 0.99$

Hence the probability that this week's production will be between 400 and 600 is atleast 0.99.

Example:

Prove that in 2000 throws with a coin the probability that the number of heads lies between 900 and 1100 is at least $19/20$

Soln

Let X be the number of heads. Here X follows binomial distribution with probability of success = probability of getting head = $1/2$.

$\therefore X \sim Bin(n, \theta)$, where $n = 2000$ & $\theta = 1/2$.

Now mean, $\mu = E(X) = n\theta = (2000)\left(\frac{1}{2}\right) = 1000$

Variance, $\sigma^2 = V(X) = n\theta(1-\theta)$

$= (2000)\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = 500$

Now $P(900 < X < 1100) = P(900 - 1000 < X - 1000 < 1100 - 1000)$

$= P(-100 < X - \mu < 100)$ $[\because \mu = 1000]$

$= P[|X - \mu| < 100]$

$\geq 1 - \frac{\sigma^2}{(100)^2}$ [using remarks no. 1]

$= 1 - \frac{500}{(100)^2}$

$= 1 - \frac{1}{20} = \frac{19}{20}$

$\therefore P(900 < X < 1100) \geq \frac{19}{20}$

Hence, the probability that the number of heads lies between 900 and 1100 is atleast $19/20$.

Convergence in Probability:

Suppose $X_1, X_2, X_3, \ldots$ is a sequence of random variables. Then X_n converges in probability to a number a if for every $\epsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} P\left[|X_n - a| \geq \epsilon\right] \leq \delta \quad \forall\, n \geq n_0$$

or

$$\lim_{n\to\infty} P\left[|X_n - a| \geq \epsilon\right] \to 0$$

Notation:

$$X_n \xrightarrow{P} a$$

Weak Law of Large Numbers (WLLN):

Statement:

Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with mean μ and variance σ^2. Then the sample mean $B_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$ converges in probability to the actual mean μ. that is,

$$P(|B_n - \mu| \geq \epsilon) \to 0 \text{ as } n \to \infty \quad \forall\, \epsilon > 0.$$

Here iid = independent and $\underbrace{\text{identically distributed}}$

all have same distribution with same mean & variance.

Proof:

Given that sample mean $B_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$

Now $E(B_n) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{1}{n}\{E(X_1) + E(X_2) + \cdots + E(X_n)\}$

$= \frac{1}{n}(\mu + \mu + \cdots + \mu) = \frac{1}{n} \cdot n\mu = \mu$

$Var(B_n) = Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{1}{n^2}\{V(X_1) + V(X_2) + \cdots + V(X_n)\}$

$= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$

By Chebyshev's inequality, for any $\epsilon > 0$.

$$P(|B_n - E(B_n)| \geq \epsilon) \leq \frac{Var(B_n)}{\epsilon^2}$$

$$\Rightarrow P(|B_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

Now $\lim_{n \to \infty} P(|B_n - \mu| \geq \epsilon) \leq 0$

$\Rightarrow \lim_{n \to \infty} P(|B_n - \mu| \geq \epsilon) = 0$ $\left[\because \text{Probability cannot be less than zero}\right]$

Hence $P(|B_n - \mu| \geq \epsilon) \to 0$ as $n \to \infty$ for any $\epsilon > 0$

Therefore, B_n converges to μ in probability. "Hence proved"

Note:

(1) Here we can easily see that

$P(|B_n - \mu| < \epsilon) \to 1$ as $n \to \infty$ for any $\epsilon > 0$

(2) Weak law of large numbers is also called law of large numbers.

Strong Law of Large Numbers:

Let $\{X_n\}$ be a sequence of iid random variables with mean μ. Also let $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$

Then $\bar{X}_n$ converges almost surely to μ.

~~i.e.~~ $\cancel{P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1}$

i.e. $\lim_{n\to\infty} P(\bar{X}_n = \mu) = 1$.

Exercises:

Q. ① A random variable X with mean $\mu = 8$ and variance $\sigma^2 = 9$. Find the bound for

(a) $P(-4 < X < 20)$ [Ans → $15/16$]

(b) $P(|X-8| \geq 6)$ [Ans → $1/4$]

Q. ② No. of customers who visit a car dealer's showroom on weekend is a random variable with mean 18 and S.D. 2.5. What can be said about the probability that on a weekend the customers will be between 8 and 28?

(Ans → 0.9375 atleast)

Q. ③ Let X_i assumes the values i and $-i$ with equal probabilities. i.e. $P(X_i = i) = P(X_i = -i) = \frac{1}{2}$ and zero elsewhere. Show that the WLLN does not hold for the sequence of independent variables $X_1, X_2, X_3, \ldots$