Group Action :-

A gp action of a gp G on a set A is a map from $G \times A$ to A (written as $g \cdot a$) satisfying the following properties:

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a \quad \forall\, g_1, g_2 \in G$ & $a \in A$

(ii) $1 \cdot a = a \quad \forall\, a \in A$.

Ex:- (i) $G \times A \to A$ tirivial action.

$g \cdot a = a$

(ii) $F \times V \to V$.

$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$\alpha (r_1, r_2, \ldots, r_n) = (\alpha r_1, \alpha r_2, \ldots, \alpha r_n)$

$\alpha \in \mathbb{R}$, $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$.

Let G acts on a set A. for each fixed $g \in G$ we get a mapping.

$\sigma_g : G \to A$

as $\sigma_g(a) = g \cdot a$

(i) for each fixed $g \in G$, σ_g is a permutation of A.

(ii) the map from G to S_A defined by $g \mapsto \sigma_g$ is a homomorphism.

$S_A \to$ set of all permutation of A.

Proof:- (i) σ_g is permutation

$g \in G$, $\exists\ g^{-1} \in G$ define $\sigma_{g^{-1}} : A \to A$ as $\sigma_{g^{-1}}(a) = g^{-1}.a$

then

$$(\sigma_g \circ \sigma_{g^{-1}})(a) = \sigma_g(\sigma_{g^{-1}}(a))$$
$$= \sigma_g(g^{-1}.a) = g(g^{-1}.a)$$
$$= (g.g^{-1}).a = a$$

& $$(\sigma_{g^{-1}} \circ \sigma_g)(a) = \sigma_{g^{-1}}(\sigma_g(a))$$
$$= \sigma_{g^{-1}}(g.a) = g^{-1}(g.a)$$
$$= (g^{-1}g).a = a.$$

$\Rightarrow \sigma_{g^{-1}}$ is inverse of $\sigma_g \Rightarrow \sigma_g$ is permutation.

(ii) $\phi : G \to S_A$ as

$\phi(g) = \sigma_g$

Claim :- ϕ is homomorphism.

i.e $\phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2)$

Let $\phi(g_1 g_2)(a) = (g_1 g_2) \cdot a$

$= g_1 \cdot (g_2 \cdot a)$

$= g_1 \cdot (\sigma_{g_2})$

$= (\sigma_{g_1} \circ \sigma_{g_2})(a)$

$= (\phi(g_1) \circ \phi(g_2))(a)$

a is arbitrary.

$\Rightarrow \phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2)$

$\Rightarrow$ ϕ is a homomorphism.

Here ϕ is called the permutation representation associated to the given action

Examples:-

(i) G be gp and $A \to$ non-empty set.

$g \cdot a = a \quad \forall g \in G, \; a \in A$

then distinct elements of g induce the same permutation on A (i.e. identity permutation) and the associated permutation representation $G \to S_A$ is trivial homomorphism.

Def^n:- If G acts on a set B and distinct elements of G induce distinct permutations of B, the action is said to be faithful.

kernel of the action G on B = $\{g \in G \mid g \cdot b = b \ \ \forall b \in B\}$

Example:- (2) $G = S_A$ = symmetry group

then S_A acts on A by

$$\sigma \cdot a = \sigma(a) \qquad \forall \sigma \in S_A,\ a \in A.$$

(i) $$\sigma_1 \cdot (\sigma_2 \cdot a) = \sigma_1(\sigma_2(a))$$
$$= (\sigma_1 \circ \sigma_2) \cdot a =$$

(ii) $$\sigma_e \cdot a = \sigma_e(a) = a$$

The associated permutation representation is the identity map from S_A to S_A.

Example (3.) Consider D_{2n} - (~~to~~ vertices of n-gon).

[In Gallian, we represent it by D_n]

The map $D_{2n} \times \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ by

$$\alpha \cdot i \ \ \cancel{(\alpha, i)} \mapsto \sigma_\alpha(i)$$ (σ_α is symmetry by the way α permutes the corresponding vertices)

is a action.

(i) let $\alpha, \beta \in D_{2n}$

then $$\alpha \cdot (\beta \cdot i) = \alpha \cdot (\sigma_\beta(i))$$

$= \alpha \cdot j$ (let $\sigma_\beta(i) = j$)

$= \sigma_\alpha(j) = k$

Now $(\alpha\beta) \cdot i = \sigma_{\alpha\beta}(i)$

$= (\sigma_\alpha \circ \sigma_\beta) i = k$.

(ii) $e \cdot i = \sigma_e(i) = i \quad \forall i \in \{1, 2, \ldots, n\}$

This action is faithful,

let $\phi : D_{2n} \to S_{D_{2n}}$ defined as

$$\phi(\alpha) = \sigma_\alpha$$

To show ϕ is faithful, we need to show that ϕ is one-one.

i.e. $\phi(\alpha) = \phi(\beta)$

$\Rightarrow \sigma_\alpha = \sigma_\beta$

$\Rightarrow \sigma_\alpha \circ \sigma_{\beta^{-1}} = \sigma_e$

$\Rightarrow \sigma_{\alpha\beta^{-1}} = \sigma_e$

$\Rightarrow \alpha\beta^{-1} = e \Rightarrow \boxed{\alpha = \beta.}$

Now let $n = 3$, then D_6 has order 6 and S_3 has order 6. and we have an injective homomorphism from D_6 to S_3.

$\therefore D_6 \cong S_3$.

(In general $D_{2n} \ncong S_n$).

Ex(5) G be any gp and $A = G$.

Define a map from $G \times A \longrightarrow A$ as

$$g \cdot a = ga$$

is an action (left regular action).

this is faithful.

Let $\phi : G \longrightarrow S_A$ be homomo

$$\phi(g) = \sigma_g$$

$$\phi(g_1) = \phi(g_2)$$

$\Rightarrow \quad \sigma_{g_1} = \sigma_{g_2}$

$\Rightarrow \quad \sigma_{g_1}(a) = \sigma_{g_2}(a)$

$\Rightarrow \quad g_1 a = g_2 a$

$\Rightarrow \quad \boxed{g_1 = g_2}$

Centralizers and Normalizers

Let A be any non-empty subset of gp G.

then

Defn:- Define $C_G(A) = \{ g \in G \mid gag^{-1} = a \ \forall \ a \in A \}$. This subset of G is called the centralizer of A in G. $C_G(A)$ is the set of elements of G which commute with every element of A.

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$C_G(A)$ is a subgp of G.

Clearly, $1 \in C_G(A) \Rightarrow C_G(A) \neq \phi$.

Let $x, y \in C_G(A) \Rightarrow xax^{-1} = a$ & $yay^{-1} = a$.

Claim :- $xy \in C_G(A)$

$$(xy)a(xy)^{-1} = xyay^{-1}x^{-1}$$

$$= xax^{-1} = a$$

$\Rightarrow xy \in C_G(A)$

Now let $x \in C_G(A)$, Claim :- $x^{-1} \in C_G(A)$

$\cancel{x^{-1}ax} \; \cancel{ax^{-1}x} \Rightarrow ax = xa$

$xax^{-1} = a \Rightarrow a = x^{-1}ax$

$\Rightarrow C_G(A) \leq G$

Defn :- Define $Z(G) = \{g \in G \mid gx = xg \; \forall x \in G\}$, the set of elements commuting with all the elements of G. Called center of G.

Defn :- Define $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. Define the normalizer of A in G to be the set

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}.$$

Claim ① $N_G(A)$ is a subgp of G.

② $C_G(A) \le N_G(A)$.

① is same as $C_G(A)$.

Now we show $C_G(A) \le N_G(A)$.

let $x \in C_G(A)$

$\Rightarrow \quad xax^{-1} = a \quad \forall\, a \in A.$

$\Rightarrow \quad xAx^{-1} = A$

$\Rightarrow \quad x \in N_G(A).$

$\Rightarrow \quad C_G(A) \le N_G(A).$

Example:- let $G = D_8$

$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

and let $A = \{1, r, r^2, r^3\}$

Claim:- $C_G(A) = A$.

Since all powers of r commute with each other

$\Rightarrow \quad A \le C_G(A).$

Since, $sr \ne rs \Rightarrow s \notin C_G(A)$.

And if sr^i $(i = 1, 2, 3)$ is in $C_G(A)$, then

$4 < |C_G(A)| < 8$ (Not possible).

$\therefore \quad C_G(A) = A.$

Claim:- $N_{D_8}(A) = D_8$

firstly, $A \leq N_{D_8}(A)$

and $sAs^{-1} = \{s1s^{-1}, srs^{-1}, sr^2s^{-1}, sr^3s^{-1}\}$

$= \{1, r^3, r^2, r\} = A$

$\Rightarrow s \in N_{D_8}(A)$

$\Rightarrow N_{D_8}(A) = D_8$.

Claim:- $Z(D_8) = \{1, r^2\}$

firstly, $Z(G) \leq C_G(A) \quad \forall A \leq G$.

$\therefore Z(D_8) \leq C_G(A) = A = \{1, r, r^2, r^3\}$

As $sr \neq rs \Rightarrow r \notin Z(D_8)$

IIIly $sr^3 \neq r^3 s \Rightarrow r^3 \notin Z(D_8)$

$\Rightarrow Z(D_8) \leq \{1, r^2\}$

and $r^2 s = sr^2$

$r^2(sr^i) = r^2 sr^i = sr^{i+2}$

& $(sr^i)r^2 = sr^{i+2}$

$\Rightarrow r^2 \in Z(D_8)$

$\Rightarrow Z(D_8) = \{1, r^2\}$.

Quaternion Group :-

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with product . computed as.

$$1 \cdot a = a \cdot 1 = a \quad \forall \ a \in Q_8 .$$

$$(-1).(-1) = 1 \quad , \quad (-1).a = a.(-1) = -a \quad \forall \ a \in Q_8$$

$$i \cdot i = j \cdot j = k \cdot k = -1$$

$$i \cdot j = k \qquad j \cdot i = -k$$

$$j \cdot k = i \qquad k \cdot j = -i$$

$$k \cdot i = j \qquad i \cdot k = -j$$

	1	−1	i	−i	j	−j	k	−k
1	1	−1	i	−i	j	−j	k	−k
−1	−1	1	−i	i	j	j	−k	k
i	i	−i	−1	1	k	−k	−j	j
−i	−i	i	1	−1	−k	k	j	−j
j	j	−j	−k	k	−1	1	i	−i
−j	−j	j	k	−k	1	−1	−i	i
k	k	−k	j	−j	−i	i	−1	1
−k	−k	k	−j	j	i	−i	1	−1

Check this forms a grp ✓.

$$C_{Q_8}(i) = \{\pm 1, \pm i\}$$

$$Z(Q_8) = \{1, -1\}$$

ool

Stabilizers and Kernels of Group Action

If G is a gp acting on a set S and s is some fixed element of S, then stabilizer of s in G is

$$G_s = \{g \in G \mid g.s = s\}$$

Claim :- $G_s \leq G$.

As $1 \in G_s \Rightarrow G_s \neq \phi$

Let $x, y \in G_s \Rightarrow x.s = s$ & $y.s = s$

$$(xy).s = x.(y.s)$$
$$= x.s = s$$

$\Rightarrow xy \in G_s$.

also, $s = 1.s = (y^{-1}y).s$
$$= y^{-1}.(y.s)$$
$$= y^{-1}.(s)$$

$\Rightarrow y^{-1}.(s) = s$

$\Rightarrow y^{-1} \in G_s \quad \forall\, y \in G_s$.

$\Rightarrow \boxed{G_s \leq G}$

Kernel of action $= \{g \in G \mid g.s = s \;\forall\, s \in S\} \leq G$.

(Check)

Result!- ker of Group Actions $\subseteq G_s$ $(\forall s \in S)$.
or $= \bigcap_{s \in S} G_s$

Ex!- ① $G = S_n$ $A = \{1, 2, \ldots, n\}$, G acts on A as
$\sigma . i = \sigma(i)$
then associated permutation representation is
$\phi : S_n \to S_n$ as $\phi(\sigma) = \sigma$.

Now, $G_i = \{\sigma \in S_n \mid \cancel{\sigma(i)}\ \sigma . i = i\}$

then $|G_i| = (n-1)!$ & $G_i \cong S_{n-1}$ (Check?).

and $\ker G^A = \{\sigma \in S_n \mid \sigma . i = i \ \forall i \in A\}$
$=$ Identity map.

② Suppose $D_8 = G$ acts on $A = \{1, 2, 3, 4\}$
as $(\alpha, i) \to \sigma_\alpha(i)$

$G_1 = \{\alpha \in G \mid (\alpha, 1) = 1\}$. this is $\forall a \in A$.
$= \{1, t\}$. $1 = R_0$

where t is the reflection about line of symmetry passing through vertex 1 and center of square.

and $\ker G^A = \{\alpha \in G \mid (\alpha, a) = a \ \forall a \in A\}$
$= 1$ = identity map.

Ex: 3

Centralizers, normalizers and centers of groups are subgroups is a special case of the fact that stablizers and kernels of actions are subgps

Let G be gp and $S = P(G)$ = collection of all subsets of G.

then G acts on S by conjugation, i.e.

$$g: B \to gBg^{-1} \quad \text{for } g \in G, \text{ and } B \subseteq G \ (\in P(G))$$

(i) $g_1 \cdot (g_2 \cdot B) = g_1 \cdot (g_2 B g_2^{-1})$

$$= g_1 g_2 B g_2^{-1} g_1^{-1} = g_1 g_2 B (g_1 g_2)^{-1}$$

$$= (g_1 g_2) \cdot B.$$

(ii) $1 \cdot B = 1 B 1^{-1} = B.$

$\Rightarrow$ Conjugation is an action.

Now let $A \in P(G)$ $(A \subseteq G)$

then $G_A = \{ g \in G \mid g \cdot A = A \}$

$$= \{ g \in G \mid gAg^{-1} = A \}$$

$\Rightarrow$ $G_A = N_G(A)$.

As G_A is subgp of G $\Rightarrow$ $N_G(A)$ is subgp of G.

Now let $N_G(A)$ acts on set $S = A$ by conjugation

$$g \cdot a = gag^{-1}$$

(i) $g_1 \cdot (g_2 \cdot a) = g_1 (g_2 a g_2^{-1})$

$$= g_1 g_2 a g_2^{-1} g_1^{-1} = (g_1 g_2) \cdot a$$

(ii) $1 \cdot a = a$.

then ker of action $= \{ g \in N_G(A) \mid g \cdot a = a \ \forall a \in A \}$

$$= \{ g \in N_G(A) \mid gag^{-1} = a \ \forall a \in A \}$$

$$= \{ g \in N_G(A) \mid ga = ag \ \forall a \in A \}$$

$$= C_G(A) \qquad (\because C_G(A) \leq N_G(A))$$

And Now. let G acts on G by conjugation

$$g \cdot a = gag^{-1}$$

then kernel of G acting of $G = \{ g \in G \mid g \cdot a = a \ \forall a \in G \}$

$$= \{ g \in G \mid gag^{-1} = a \ \forall a \in G \}$$

$$= \{ g \in G \mid ga = ag \ \forall a \in G \}$$

$$= Z(G).$$

Ex:- (3) Let $G = D_8$ and $A = \{(1,3), (2,4)\}$

then G acts on A as

$$\alpha \cdot i = \sigma_\alpha(i)$$

Let r be the rotation of square clockwise by $\pi/2$ radian

then $r\cdot(1,3) = (2,4)$

$r\cdot(2,4) = (1,3)$

Ily $r^2\cdot(1,3) = (1,3)$

$r^2\cdot(2,4) = (2,4)$.

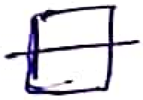
and let s be the reflection in the line which passes through 1 and 3.

then. $s\cdot(1,3) = (1,3)$

$s\cdot(2,4) = (2,4)$

and $sr\cdot(1,3) = (2,4)$

$sr\cdot(2,4) = (1,3)$

$sr \rightarrow$ 

$sr^2 =$ 

then ker of Group action on $A = \{1, r^2, s, sr^2\}$

$\Rightarrow$ Action is not faithful.

(4) D_8 on $A = \{(1,2),(3,4)\}$ is not an action.