Defn 2 $\Rightarrow$ Defn 1

Let $(L, \wedge, \vee)$ be a lattice as an algebra.
Define $a \leq b$ in L iff $a \wedge b = a$
Prove $a \wedge b = a$ iff $a \vee b = b$.

Claim $\leq$ is a partial order relation on L

i) $\forall\, a \in L$, $a \wedge a = a$ (Idempotence)
$\Rightarrow a \leq a$
$\Rightarrow \leq$ is reflexive.

ii) Let $a \leq b$ & $b \leq a$ in L
$\Rightarrow a \wedge b = a$ & $b \wedge a = b$
Since $a \wedge b = b \wedge a$
$\therefore a = b$
$\Rightarrow \leq$ is antisymmetric.

iii) Let $a \leq b$ & $b \leq c$
$\Rightarrow a \wedge b = a$ & $b \wedge c = b$
Consider $a \wedge c = (a \wedge b) \wedge c$
$= a \wedge (b \wedge c) = a \wedge b = a$.

$\therefore a \leq c$
$\Rightarrow \leq$ is transitive.
$\therefore (L, \leq)$ is a Poset.

Let $a, b \in L$.
Claim $a \wedge b = \inf\{a, b\}$ & $a \vee b = \sup\{a, b\}$.
$(a \wedge b) \vee a = a \vee (a \wedge b)$
$= a$ (By absorption)
$\Rightarrow a \wedge b \leq a$
Similarly $a \wedge b \leq b$.
$\Rightarrow a \wedge b$ is a l.b of a & b.
Let c be any other l.b of a & b.
$c \leq a$, $c \leq b$.
Then $c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c$
$\therefore c \leq a \wedge b$.
$\therefore a \wedge b$ is the glb of a & b in L
$\Rightarrow a \wedge b = \inf\{a, b\}$
$\therefore \inf\{a, b\} \in L$.

Similarly $a \vee b = \sup\{a, b\}$.

$\therefore \sup\{a, b\} \in L$

$\therefore (L, \leq)$ is a lattice as a Poset.

$\therefore$ Defn 2 $\Rightarrow$ Defn 1.

Duality Principle

Any formula or theorem involving the relations $\wedge, \vee$ & $\leq$ in a lattice L remains valid if $\wedge, \vee$ & $\leq$ are replaced by $\vee, \wedge, \geq$ everywhere in the result.

This process is called Dualising.

If $(P, \leq)$ is a Poset, then $(P, \geq)$ is also a poset.

Bounded lattice

A lattice having least element (zero element 0) & greatest element (Unit element 1) is called bounded.

ex $(\mathbb{N}, \leq)$ is not bdd.

ex A = any set

$(P(A), \subseteq, \cap, \cup)$ is a bounded lattice.

Thm Every finite lattice has a least & a greatest element.

Pf let $L = \{x_1, \longrightarrow x_n\}$ be a finite lattice.

let $y_1 = x_1$

$y_2 = y_1 \wedge x_2$

$y_3 = y_2 \wedge x_3 \ldots$ $y_k = y_{k-1} \wedge x_k$

$y_n = y_{n-1} \wedge x_n$.

Date | Page No.

<u>claim</u> y_n = least elt ~~elt~~ L.

Then $y_n = y_{n-1} \wedge x_n$
$\leq x_n$.

$y_n = y_{n-1} \wedge x_n \leq y_{n-1} = y_{n-2} \wedge x_{n-1}$
$\leq x_{n-1}$.

Similarly $y_n \leq x_k \ \forall k$

$\therefore y_n$ is the least element of L

Similarly, prove $\rightarrow$ L has a greatest elt.

<u>Sublattice</u>: A subset S of a lattice L is called a sublattice of L if S itself is a lattice w.r.t. the operations of L.

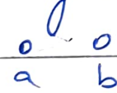
It's enough to prove that S is closed w.r.t. $\wedge$ & $\vee$ of L.

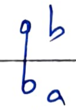
<u>Remark</u>

1) $\exists$ only one lattice of order 1.

• $0=1$

2) $\exists$ ~~only one~~ 2 posets of order 2.



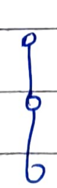


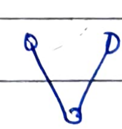
antichain | <u>chain</u> (lattice)

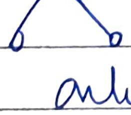
$\exists$ only one lattice of order 2.

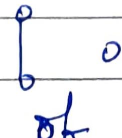
3) Posets of order 3.

(i) 

(ii) 

(iii) 

(iv) 

(V) 

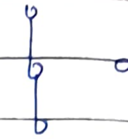
$\exists$ only one lattice of order 3 $\rightarrow$  <u>chain</u>

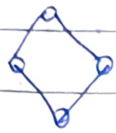
16

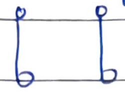
4) Draw all posets of order 4.

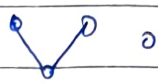
(i) 

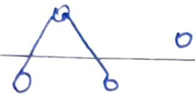
(ii) 

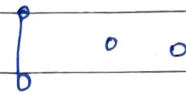
(iii) 

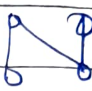
(iv) 

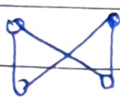
(v) 

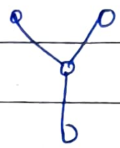
(vi) 

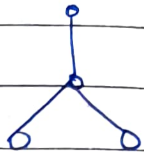
(vii) 

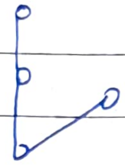
(viii) 

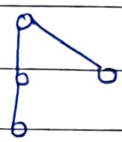
(ix) 

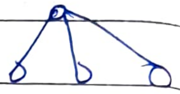
(x) 

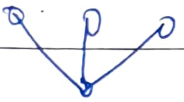
(xi) 

(xii) 

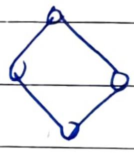
(xiii) 

(xiv) 

(xv) 

(xvi) 

$\exists$ only 2 lattices of order 4.

Chain  & rhombus 

Q Draw all posets of order 5 & lattices of order 5.