Compatible system of first-order Equations :-

Consider the following first-order partial differential eqns

$$f(x, y, z, p, q) = 0 \quad \text{—— (1)}$$

and

$$g(x, y, z, p, q) = 0 \quad \text{—— (2)}$$

then eqn no (1) and (2) are known as Compatible when every solution of eqn (1) is also a solution of eqn (2).

Condition for (1) & (2) be compatible :-

Equations (1) & (2) are compatible if

$$[f, g] = 0 ,$$

where

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} .$$

$$\frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \partial f/\partial x & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} , \quad \frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix}$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \frac{\partial g}{\partial y} & \partial g/\partial q \end{vmatrix} . \quad \frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix}$$

Example :- Show that the equations

$$xp = yq, \quad z(xp + yq) = 2xy$$

are compatible and solve them.

We may take $f(x,y,z,p,q) = xp - yq = 0$ — (1)

and $g(x,y,z,p,q) = z(xp + yq) - 2xy = 0$ — (2)

Eqn (1) & (2) are compatible if

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0 \quad \text{— (3)}$$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & zx \end{vmatrix}$$

$$= pzx - xzp + 2xy$$

$$\boxed{\frac{\partial(f,g)}{\partial(x,p)} = 2xy}$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + yq & zx \end{vmatrix}$$

$$\frac{\partial(f,g)}{\partial(z,p)} = -x^2p - xyq$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} -q & -y \\ qz - 2x & zy \end{vmatrix}$$

$$\boxed{\frac{\partial(f,g)}{\partial(y,q)} = -2xy}$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & 2y \end{vmatrix}$$

$$\boxed{\frac{\partial(f,g)}{\partial(z,q)} = +xpy + y^2 q}$$

then

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= \cancel{2xy} - x^2p^2 \cancel{- xyqp} - \cancel{2xy} + \cancel{xypq} + y^2q^2$$

$$= y^2q^2 - x^2p^2$$

$$= (yq+xp)(yq-xp)$$

$$= 0$$

{ as from eq^n (1)

$xp - yq = 0$ },

or $xp = yq$

Hence, the eq^{n}s are Compatible.

Now, solve eq^{n}s (1) & (2) for p & q,

we get

$p = y/z$ and $q = x/z$.

and

$$dz = p\,dx + q\,dy$$

$$dz = \frac{y}{z}dx + \frac{x}{z}dy$$

or

$$z\,dz = y\,dx + x\,dy$$

Integrating $\boxed{z^2 = c_1 + 2xy}$ where c_1 is a constant.

Exercise: (1) Show that the equation

$$xp - yq = x \; \& \; x^2 p + q = xz$$

are compatible and find their solution.

(2) Show that the equation $z = px + qy$ is compatible with $2xy(p^2+q^2) = z(yp + xq)$ and find their solution.

(3) Show that the eqns $f(x, y, p, q) = 0$, and $g(x, y, p, q) = 0$ are compatible if

$$\frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} = 0.$$

Verify that the eqns $p = P(x, y)$ and $q = Q(x, y)$ are compatible if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$