27 November 2020
09:30

TS

$$\frac{K}{K \cap N} \cong \frac{KN}{N}$$

SIM

$N \trianglelefteq KN$ & $\frac{KN}{N}$ well defined

$\phi : K \to \frac{KN}{N}$ such that $\phi(k) = kN \quad \forall k \in K$

ϕ homomorphism & ϕ onto

$$\phi(K) = \frac{KN}{N} \quad \text{------} \quad (1)$$

$$\ker \phi = \{k \in K \mid \phi(k) = N\}$$

$$= \{k \in K \mid kN = N\}$$

$$= \{k \in K \mid k \in N\}$$

$$= K \cap N$$

from first isomorphism theorem,

$$\frac{K}{K \cap N} \cong \phi(K)$$

& $\boxed{\frac{K}{K \cap N} \cong \frac{KN}{N}}$

(1) & (2)

Chapter 10 Q. 42 (Third Isomorphism Theorem)

If M and N are normal subgroups of G and N is subgroup of M, then

$$\frac{G/N}{M/N} \cong G/M$$

Sol^n First we shall show that $N \trianglelefteq M$

let $n \in N$ and $m \in M$

$\Rightarrow n \in N$ and $m \in G$ $\quad (\because M \trianglelefteq G)$

$\Rightarrow mnm^{-1} \in N$ $\quad (\because N \trianglelefteq G)$

$\Rightarrow N \trianglelefteq M$

$\Rightarrow M/N$ is well defined set

Now Define $\phi : G/N \to G/M$ such that

$$\phi(gN) = gM \quad \forall\, gN \in G/N \qquad \text{—— (1)}$$

(i) ϕ is well defined

let $g_1 N = g_2 N$

ϕ is O.P.

$\phi(g_1 N g_2 N)$

let $g_1 N = g_2 N$

$\Rightarrow g_2^{-1} g_1 N = N$

$\Rightarrow g_2^{-1} g_1 \in N$

$\Rightarrow g_2^{-1} g_1 \in M \quad (\because N \leq M)$

$\Rightarrow g_2^{-1} g_1 M = M$

$\Rightarrow g_1 M = g_2 M$

$\Rightarrow \phi(g_1 N) = \phi(g_2 N)$

$\therefore \phi$ is well-defined

$\phi(g_1 N g_2 N)$

$= \phi(g_1 g_2 N)$

$= g_1 g_2 M$

$= (g_1 M)(g_2 M)$

$= \phi(g_1 N) \phi(g_2 N)$

$\therefore \phi$ is O.P.

Now, for every $gM \in G/M$, $\exists gN \in G/N$ such that

$$\phi(gN) = gM$$

$\therefore \phi$ is onto $\Rightarrow \phi(G/N) = G/M$ (2)

Now, $\ker \phi = \{ gN \in G/N \mid \phi(gN) = M \}$

$= \{ gN \in G/N \mid gM = M \}$

$= \{ gN \in G/N \mid g \in M \}$

$= \{ gN \in M/N \}$

$g \in G \Rightarrow gN \in G/N$

$g \in M \Rightarrow gN \in M/N$

$$= M/N$$

Now, from first isomorphism theorem,

$$\frac{G/N}{M/N} \cong \phi(G/N)$$

$$\Rightarrow \boxed{\frac{G/N}{M/N} \cong G/M}$$ (from eqn 2)

$a \longrightarrow$ fixed.

$|a| = |a^k| = n$ iff $\gcd(k, n) = 1$

$|3| = 11$

$3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9, 3^{10}$

3 6 9 12 15 18 21 24 27 30