Subgroups

(Defn) let G be a group and let H be a (non-empty) subset of G.
then H is said to be subgroup of G iff H is also a Group w.r.t the same operation.

(Remark) (i) G & $\{e\}$ are always subgroup of G.
(ii) H is called proper subgroup if $H \neq G$
(iii) H is called non-trivial subgroup if $H \neq \{e\}$
(iv) $\{e\}$ is called trivial subgroup of G.

(Notation) H is subgroup of G,
we write $H \leq G$.

Theorem (1) (conditions for subgroup) let H be a non-empty subset of a group G. A necessary and sufficient condition for H to be subgroup of G that is
(i) $a, b \in H \Rightarrow ab \in H \quad \forall\, a, b \in H$
(ii) $a \in H \Rightarrow a^{-1} \in H \quad \forall\, a, b \in H$

(proof) First suppose H is subgp of G.
$\therefore$ H is itself a group.
$\therefore$ the conditions (i) & (ii) must hold.

Conversely, let (i) & (ii) holds
Now H is non-empty $\therefore a \in H$
$\therefore a^{-1} \in H$ [by (ii)]
As a & $a^{-1} \in H$ $\therefore aa^{-1} \in H$ [by (i)]
$\Rightarrow e \in H$
Further G is a group
$\therefore (ab)c = a(bc) \;\forall\, a, b, c \in G$
In particular, $(ab)c = a(bc) \;\forall\, a, b, c \in H$ [$\because H \subseteq G$]
Thus, H is itself a group
and Hence H is subgroup of G.

Theorem ② (Single criteria for subgroup) Let H be a non-empty subset of a group G. A necessary and sufficient condition for H to be a subgroup of G is that $a, b \in H \Rightarrow ab^{-1} \in H \quad \forall\, a, b \in H$

proof) First suppose H is subgroup of G
$\therefore$ H is itself a group.
let $a, b \in H$
As $b \in H \Rightarrow b^{-1} \in H$ (by inverse property)
As $a, b^{-1} \in H \therefore ab^{-1} \in H$ (By closure property)
$a, b \in H \Rightarrow ab^{-1} \in H$ — ①

Conversely, let $a, b \in H \Rightarrow ab^{-1} \in H$
to show: H is subgroup of G

Step (i) As H is non-empty $\therefore \exists\, a \in H$
As $a, a \in H \therefore aa^{-1} \in H$ (by ①)
i-e $e \in H$

Step (ii) let $a \in H$
As $e, a \in H \therefore ea^{-1} \in H$ (by ①)
i-e $a^{-1} \in H$

Step (iii) let $a, b \in H$
As $b \in H \therefore b^{-1} \in H$ (by step (ii))
As $a, b^{-1} \in H \therefore a(b^{-1})^{-1} \in H$ (by ①)
i-e $ab \in H$

Step (iv) G is a group, we have
$(ab)c = a(bc) \quad \forall\, a, b, c \in G$
In particular $(ab)c = a(bc) \quad \forall\, a, b, c \in H$
$[\because H \subseteq G]$

Thus, H is itself a group
Hence H is subgroup of G.