(6.5) ONE TO ONE AND LINEAR TRANSFORMATIONS

(Definition) let V and W be vector space and let $T : V \to W$ be linear transformation (L.T)

① T is one to one
if for all $v_1, v_2 \in V$, $T(v_1) = T(v_2)$
$\Rightarrow v_1 = v_2$

② T is onto
if for every $w \in W$, there is some $v \in V$
s.t $T(v) = w$

(Example) let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be L.T
given by $T([x, y]) = [x, -y]$ — ①
Show T is one to one & onto

(Solution) let $T([x_1, y_1]) = T([x_2, y_2])$ where $[x_1, y_1] \in \mathbb{R}^2$, $[x_2, y_2] \in \mathbb{R}^2$
$\Rightarrow [x_1, -y_1] = [x_2, -y_2]$ [Using ①]
$\Rightarrow x_1 = x_2$ & $y_1 = y_2$
$\therefore$ T is one to one

T is onto, since for every $[x, y] \in \mathbb{R}^2$
there is vector $[x, -y] \in \mathbb{R}^2$ s.t $T([x, -y]) = [x, y]$

(Example) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be L.T
given by $T([x_1, x_2, x_3]) = [x_1, x_2, 0]$
show that T is neither one to one nor onto.

(Solution) consider $v_1 = [1, 1, 0] \in \mathbb{R}^3$
& $v_2 = [1, 1, 1] \in \mathbb{R}^3$

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Also, $T(v_1) = [1,1,0]$
$T(v_2) = [1,1,0]$ using $T([x_1,x_2,x_3]) = [x_1,x_1,0]$

Here $T(v_1) = T(v_2)$

But $v_1 \neq v_2$

$\therefore T$ is Not one to one.

T is Not onto, since there is vector $w = [0,0,1] \in \mathbb{R}^3$ but there is no vector $v = [x_1,x_2,x_3] \in \mathbb{R}^3$

s.t $T(v) = w$

$\therefore T$ is Not onto.

Theorem (6.12) let V and W be vector spaces and let $T: V \to W$ be L.T then T is one to one $\Leftrightarrow$ $Ker(T) = \{0_V\}$

(proof) Let T is one to one
to show! $Ker(T) = \{0_V\}$ where 0_V is zero vector of V.

Since $Ker(T)$ is subspace
$\Rightarrow \{0_V\} \subseteq Ker(T)$ — ①

Let $v \in Ker(T)$
$\Rightarrow T(v) = 0_W = T(0_V)$ (where 0_W is zero vector of W)
$\Rightarrow v = 0_V$ ($\because$ T is 1-1)
$\Rightarrow Ker(T) \subseteq \{0_V\}$ — ②

from ① & ②
$Ker(T) = \{0_V\}$

Conversely let $Ker(T) = \{0_V\}$
to show! T is 1-1 (one to one)
let $v_1, v_2 \in V$ s.t $T(v_1) = T(v_2)$
$\Rightarrow T(v_1 - v_2) = T(v_1) - T(v_2) = 0_W$
$\Rightarrow v_1 - v_2 \in Ker(T)$ (using definition of KerT)
As $Ker(T) = \{0_V\}$
$\Rightarrow v_1 - v_2 = 0_V$
$\Rightarrow v_1 = v_2$ $\Rightarrow$ T is one to one.

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Corollary (6.13) If $m > n$, then there is No one to one L.T from $\mathbb{R}^m \to \mathbb{R}^n$.

(proof) let $T: \mathbb{R}^m \to \mathbb{R}^n$ be L.T

By Dimension theorem, we have

$$\dim(\ker(T)) = \dim(\mathbb{R}^m) - \dim(\text{range}(T))$$
$$= m - \dim(\text{range}(T))$$
$$\geq m - n > 0$$

$\Rightarrow \dim(\ker(T)) > 0$

$\Rightarrow \ker(T) \neq \{0\}$

By using above theorem

$\Rightarrow$ T is Not one to one.

Theorem (6.14) Let V & W be vector space and let $T: V \to W$ be L.T, then T is onto if and only if If W is finite dimensional, then T is onto if and only if

$\dim(\text{range}(T)) = \dim(W)$

(proof) If T is onto $\Leftrightarrow$ range(T) = W

Since W is finite dimensional & range(T) is subspace of W

$\therefore$ By theorem 5.25

[let V be finite - dimensional V.S & let W be subspace of V then W is also finite dimensional with

$\dim(W) \leq \dim(V)$

Moreover $\dim(W) = \dim(V) \Leftrightarrow W = V$]

$\therefore$ Using theorem 5.25

$\dim(\text{range}(T)) = \dim(W) \Leftrightarrow \text{range}(T) = W$

Hence Proved.

Corollary (6.15) If $m > n$, then there is no onto L.T from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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(proof)

let $T: \mathbb{R}^n \to \mathbb{R}^m$ be L.T

By dimension theorem

$\dim(\text{range}(T)) = \dim(\mathbb{R}^n) - \dim(\ker(T))$

$= n - \dim(\ker(T))$

$\leq n$

$< \dim(\mathbb{R}^m)$

$\Rightarrow \dim(\text{range}(T)) < \dim(\mathbb{R}^m)$

$\therefore$ By theorem 6.14, T is not onto.

Example (43) let $T: M_{2\times3} \to M_{2\times2}$ be L.T

given by $T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+b & a+c \\ d+e & d+f \end{bmatrix}$

Show that T is onto but Not one to one.

(Solution) let $\begin{bmatrix} b & c \\ e & f \end{bmatrix} \in M_{2\times2}$

then there is a matrix $\begin{bmatrix} 0 & b & c \\ 0 & e & f \end{bmatrix} \in M_{2\times3}$

s.t $T\left(\begin{bmatrix} 0 & b & c \\ 0 & e & f \end{bmatrix}\right) = \begin{bmatrix} b & c \\ e & f \end{bmatrix}$ $\Rightarrow$ T is onto

$\therefore$ by theorem 6.14

$\dim(\text{range}(T)) = \dim(M_{2\times2}) = 4$

$\therefore$ By Dimension theorem

$\dim(\ker(T)) = \dim(M_{2\times3}) - \dim(\text{range}(T))$

$= 6 - 4$

$= 2$ $\Rightarrow \ker(T) \neq \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$

By theorem 6.12

T is Not one to one.

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Theorem 6.16 Let V & W be V.S and let $T: V \to W$ be one to one L.T
If S is L.I (linearly independent) subset of V
then $T(S)$ is L.I in W.

(proof) Suppose $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is finite subset of $T(S)$, where $v_1, v_2, \dots, v_n \in S$
Let $a_1, a_2, \dots a_n \in \mathbb{R}$ s.t

$a_1 T(v_1) + a_2 T(v_2) + \dots + a_n T(v_n) = 0_W$

$\Rightarrow T(a_1 v_1 + \dots + a_n v_n) = 0_W$

$\Rightarrow a_1 v_1 + \dots + a_n v_n \in Ker(T)$

As T is 1-1 (one to one)
By theorem 6.12
we have $Ker(T) = \{0_V\}$

$\Rightarrow a_1 v_1 + \dots + a_n v_n = 0_V$

Since, S is L.I, we have $a_1 = a_2 = \dots a_n = 0$

$\Rightarrow \{T(v_1), T(v_2), \dots, T(v_n)\}$ is L.I

Theorem (6.17) Let V & W be V.S and let $T: V \to W$ be onto linear transformation
If any subset S of V spans V
then $T(S)$ spans W.

(proof) Suppose S spans V
to show: $T(S)$ spans W
Since T is onto, there is a vector $v \in V$
s.t $T(v) = w$ where $w \in W$
Since S spans V
there exists vectors $v_1, v_2, \dots v_n \in V$
& scalars $a_1, a_2, \dots, a_n \in \mathbb{R}$

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Soit
$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

Then $w = T(v) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$

(∵ T is L.T)

$\Rightarrow w = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$

$\Rightarrow T(S)$ Spans W.

(6.6) ISOMORPHISMS

(Definition) Invertible L.T

Let V & W be V.S & let $T: V \to W$ be Linear transformation. Then T is said to be invertible Linear transformation if there is a function $S: W \to V$ s.t $(S \circ T)(v) = v$ for all $v \in V$

& $(T \circ S)(w) = w$ for all $w \in W$

Such a function S is called Inverse of T.

Remark ① If a L.T (linear transformation) $T: V \to W$ is invertible, its inverse is unique & is denoted by T^{-1}.

Remark ②. If $T: V \to W$ is invertible L.T then $T^{-1}(T(v)) = v$ for all $v \in V$

& $T^{-1}(T(w)) = w$ for all $w \in W$.

Definition Isomorphism

Let V & W be V.S.

A L.T $T: V \to W$ is both 1-1 (one to one) & onto is called an Isomorphism of V onto W.

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Example (48)

Show that a mapping $L: P_2 \to P_2$ given by $L(p(x)) = p(x) + p'(x)$ is isomorphism

where P_2 is V.S of all polynomials of degree ≤ 2

(Solution) (1) To show: L is Linear operator

(2) To show: L is 1-1

(3) L is onto

Let $p(x)$ & $q(x) \in P_2$

$$L(p(x) + q(x)) = (p(x) + q(x)) + (p(x) + q(x))' \quad \left[\text{using } L(p(x)) = p(x) + p'(x)\right]$$

$$= p(x) + q(x) + p'(x) + q'(x)$$

$$= (p(x) + p'(x)) + (q(x) + q'(x))$$

$$= L(p(x)) + L(q(x))$$

Let $\alpha \in \mathbb{R}$ & $p(x) \in P_2$, we have

$$L(\alpha p(x)) = ((\alpha)p(x)) + (\alpha(p(x)))')$$

$$= \alpha p(x) + \alpha p'(x)$$

$$= \alpha (p(x) + p'(x))$$

$$= \alpha L(p(x))$$

$\therefore$ L is Linear operator

To show: L is 1-1 (one to one)

$Ker(L) = \{ax^2 + bx + c \,;\, L(ax^2 + bx + c) = 0\}$

where polynomial $ax^2 + bx + c \in P_2$

$= \{ax^2 + bx + c \,;\, ax^2 + (2a+b)x + b + c = 0\}$

$= \{ax^2 + bx + c \,;\, a = 0,\ 2a + b = 0,\ b + c = 0\}$

we get $a = 0,\ b = 0,\ c = 0$

$\therefore Ker(T) = \{0\} \Rightarrow L$ is 1-1 (by theorem 6.12)

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L is onto

For any polynomial $w = p(x) = ax^2 + bx + c \in P_2$

there is polynomial

$v = f(x) = ax^2 + (b-2a)x + (c-b+2a) \in P_2$

s.t $L(v) = ax^2 + (b-2a+2a)x + (b-2a+(c-b+2a))$

$= ax^2 + bx + c = w$

$\therefore$ L is onto $\Rightarrow$ L is Isomorphism.

Theorem (6.18) Let $T: V \to W$ be L.T then T is an Isomorphism if & only if T is an invertible L.T. Moreover if T is invertible then T^{-1} is also L.T

(proof) let $T: V \to W$ be L.T

& $T^{-1}: W \to V$ is its inverse

then $T^{-1}(T(v)) = v$ for all $v \in V$

& $T^{-1}(T(w)) = w$ for all $w \in W$

to show! T is isomorphism (i.e T is 1-1 & onto)

let $T(v_1) = T(v_2)$ for $v_1, v_2 \in V$

$\Rightarrow T^{-1}(T(v_1)) = T^{-1}(T(v_2))$

$\Rightarrow v_1 = v_2$

$\therefore$ T is 1-1

(to prove) T is onto

Let $w \in W$

Now T^{-1} maps w to vector v of V

i.e $T^{-1}(w) = v$

$\Rightarrow T(v) = T(T^{-1}(w)) = w$

$\therefore$ T is onto

Hence T is Isomorphism

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Conversly

let $T: V \to W$ is L.T, 1-1 & onto

to show: T has inverse $S: W \to V$

let $w \in W$

Since T is onto

$\therefore w \in W$, we have $T(v) = w$

Since T is 1-1 (one to one)

v is unique vector in V s.t $T(v) = w$

Now $S: W \to V$ defined by $S(w) = v$

where v is unique vector in V s.t $T(v) = w$

Also, clearly S is well defined

$(T \circ S)(w) = T(S(w)) = T(v) = w$

And $(S \circ T)(v) = S(T(v)) = S(w) = v$

$\therefore T$ is invertible, with inverse T^{-1}

to show! T^{-1} is L.T

Let $w, w_1, w_2 \in W$ & $c \in \mathbb{R}$

consider, ~~$T(T^{-1}(w_1 + w_2))$~~ $T(T^{-1}(w_1) + T^{-1}(w_2)) = T(T^{-1}(w_1)) + T(T^{-1}(w_2))$ ($\because T$ is L.T)

$= w_1 + w_2$ (by defination of T^{-1})

$= T\, T^{-1}(w_1 + w_2)$ (definition of T^{-1})

Also, $T(c\,(T^{-1}(w))) = c\,T(T^{-1}(w))$ ($\because T$ is L.T)

$= c\,w$ (by definition of T^{-1})

$= T(T^{-1}(c\,w))$ (by definition of T^{-1})

$\therefore T^{-1}$ is L.T

(Hence Proved)

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Theorem (6.19)

let V & W be two non-trivial finite dimensional V.S with ordered bases B and C, respectively
let $T: V \to W$ be L.T, then T is Isomorphism if and only if matrix representation A_{BC} for T with respect to B and C is non-singular.

Example (48) let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be L.O (linear operator) defined by $T(v) = Av$, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Is T is Isomorphism

(Solution) Since $|A| = 1$
i.e $|A| \neq 0$
$\therefore$ By theorem 6.19
T is Isomorphism

Theorem (6.20) suppose V, W & X are V.S & suppose that $T_1: V \to W$ and $T_2: W \to X$ are both Isomorphism then composition $T_2 \circ T_1: V \to X$ is Also Isomorphism.

(Definition) Isomorphic Vector Space

Let V & W be V.S, we say V is Isomorphism to W, write as $V \cong W$
if there exists an isomorphism $T: V \to W$

Theorem (6.21) Properties of Isomorphism

1. Every V.S V is isomorphism to itself.
2. If $V \cong W \Rightarrow W \cong V$
3. If $V \cong W$ & $W \cong X \Rightarrow V \cong X$.

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Theorem (6.22)

Suppose $V \cong W$ and V is finite-dimensional. Then W is finite dimensional and $\dim V = \dim W$.

(proof) Since $V \cong W$ then $T : V \to W$ is isomorphism.

let $\dim(V) = n$

and $B = \{v_1, v_2, \ldots, v_n\}$ be basis for V.

Show: $T(B) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is basis for W.

let $c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) = 0_W$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$

$\Rightarrow T(c_1 v_1 + \cdots + c_n v_n) = 0_W$ ($\because T$ is L.T)

Since T is onto

$\therefore c_1 v_1 + \cdots + c_n v_n = 0_V$ ($\because 0_W = T(0_V)$)

Also B is L.I

we get $c_1 = c_2 = \cdots = c_n = 0$

$\therefore T(B)$ is L.I

Show: $T(B)$ spans W.

let $w \in W$

Since T maps V onto W

there there exists a vector $v \in V$ s.t $T(v) = w$ — (1)

Also, B is basis for V

$\therefore B$ spans V

there exists scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$ s.t

$v = a_1 v_1 + \cdots + a_n v_n$

$\Rightarrow T(v) = T(a_1 v_1 + \cdots + a_n v_n)$

$\Rightarrow w = a_1 T(v_1) + \cdots + a_n T(v_n)$ (Using (1) & T is L.T)

$\therefore T(B)$ spans W

thus $T(B)$ is basis for W containing exactly n vectors.

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Hence $\dim W = n$

$\therefore \dim(V) = \dim(W)$

Theorem (6.23) If V is any n-dimensional vector space then $V \cong R^n$

(proof) Suppose that $\dim V = n$

let $B = \{v_1, v_2, \ldots, v_n\}$ be ordered basis for V.

Define $T : V \to R^n$ as follows;

$$T(v) = [v]_B$$

where for every vector $v \in V$ has its co-ordinatization $[v]_B$ with respect to B.

to show; V is isomorphic to R^n

Let $v_1, v_2 \in V$

then $T(v_1 + v_2) = [v_1 + v_2]_B$

$= [v_1]_B + [v_2]_B$ (because $[v]_B$ with respect to B)

$= T(v_1) + T(v_2)$

let $\alpha \in R$.

consider $T(\alpha v) = [\alpha v]_B$

$= \alpha [v]_B$

$= \alpha T(v)$ for any $\alpha \in R$, $v \in V$

$\therefore T$ is L.T

let $v \in Ker(T)$

$\Leftrightarrow T(v) = [0, 0, \ldots, 0]$

$\Leftrightarrow [v]_B = [0, 0, \ldots, 0]$

$\Leftrightarrow v = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$

$\Leftrightarrow v = 0$

Hence $Ker(T) = \{0_V\}$

$\therefore T$ is one to one.

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Let $a = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^n$

then $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in V$ s.t

$T(v) = [v]_B = [a_1, a_2, \ldots, a_n] = a$

$\therefore$ T is onto.

Hence $V \cong \mathbb{R}^n$

Corollary: (6.24) Any two finite-dimensional V.S V and W are isomorphic if they have the same dimension. If $\dim V = \dim W$ then $V \cong W$

(proof) Let $\dim V = \dim W = n$

By theorem 6.23

$V \cong \mathbb{R}^n$ & $W \cong \mathbb{R}^n$

$\Rightarrow V \cong W.$