

Beta and Gamma functions

Beta function: Beta function is the Eulerian Integral of first kind, denoted by

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

Remarks:

$$\beta(1, 1) = 1$$

Gamma function: Gamma function is the Eulerian Integral of second kind, denoted by

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad ; \quad n > 0$$

Remarks:

$$\Gamma(1) = 1$$

Some properties of Beta function:

$$1) \quad \beta(m, n) = \beta(n, m)$$

$$2) \quad \beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

$$3) \quad \beta(m, n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

$$4) \quad 2\beta(m, n) = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

$$5) \quad \beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \quad ; \quad \begin{matrix} m > -1 \\ n > -1 \end{matrix}$$

$$6) \quad \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2 \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta \quad ; \quad \begin{matrix} p > -1 \\ q > -1 \end{matrix}$$

$$7) \quad \beta(m, n) = \beta(m+1, n) + \beta(m, n+1) \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

Ex 1. consider the integral

$$I = \int_0^1 x^{\frac{1}{2}} (1-x)^4 dx \quad \text{can be written as}$$

$$= \int_0^1 x^{\frac{3}{2}-1} (1-x)^{5-1} dx$$

$$= \beta\left(\frac{3}{2}, 5\right)$$

Ex 2:-

consider the integral

$$I = \int_0^1 x^{\frac{1}{2}} (1-x)^{-3} dx \quad \text{not beta function as}$$

$$n-1 = -3$$

$$\Rightarrow n = -3+1 = -2 \quad (\text{negative}).$$

Properties

①

$$\beta(m, n) = \beta(n, m) \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

Sol:-

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

put $1-x = y$
 $dx = -dy$

$$\therefore \beta(m, n) = - \int_1^0 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy$$

$$= \beta(n, m)$$

$$\therefore \boxed{\beta(m, n) = \beta(n, m)}$$

②

$$\beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy \quad ; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

Sol:-

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

put $x = \frac{y}{1+y}$ i.e. $y = \frac{x}{1-x}$

$$dx = \frac{1}{(1+y)^2} dy$$

$$\leftarrow 1-x = \frac{x}{y} = \frac{y}{1+y} \cdot \frac{1}{y}$$

$$(1-x) = \frac{1}{1+y} \quad \text{--- (1)}$$

$$\therefore \beta(m, n) = \int_0^{\infty} \left(\frac{y}{1+y} \right)^{m-1} \left(\frac{1}{1+y} \right)^{n-1} \frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}} dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Hence

$$\boxed{\beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy}$$

$$x = \frac{y}{1+y} \Rightarrow y = x(1-x)$$

$$y = x \cdot \frac{1}{1-x}$$

using (1)

3) $\beta(m, n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$; $m > 0$
 $n > 0$

Sol:-

we know that $\beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$

$$\beta(m, n) = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy + \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy \quad \text{--- (1)}$$

$I_1 \qquad \qquad \qquad I_2$

where $I_2 = \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$

Now put $y = \frac{1}{x}$
 $dy = -\frac{1}{x^2} dx$

$$\therefore I_2 = - \int_1^0 \left(\frac{1}{x}\right)^{m-1} \cdot \frac{1}{\left(1+\frac{1}{x}\right)^{m+n}} \cdot \frac{1}{x^2} dx$$

$$= \int_0^1 \frac{x^{m+n}}{(1+x)^{m+n}} \cdot \frac{1}{x^{m+1+2}} dx$$

$$= \int_0^1 \frac{x^{m+n-m+1-2}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

or we can write (if we put $x=y$)

$$I_2 = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$\therefore$ Eqn (1) becomes

$$\boxed{\beta(m, n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy}$$

$$(4) \quad 2\beta(m, n) = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

we know that

sol:-

$$\beta(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy \quad \text{--- (1)}$$

put or replace $n \leftrightarrow m$

$$\beta(n, m) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\text{or } \beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy \quad \text{--- (2)}$$

(since $\beta(n, m) = \beta(m, n)$)

adding (1) & (2) we get

$$2\beta(m, n) = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

(3)

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \quad ; \begin{matrix} m > -1 \\ n > -1 \end{matrix}$$

sol:-

$$\text{we know that } \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{put } x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$$

$$\therefore \beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-2} \theta \cos^{2n-2} \theta \sin \theta \cos \theta d\theta$$

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

put

$$2m-1 = p \Rightarrow m = \frac{p+1}{2}$$

$$\& 2n-1 = q \Rightarrow n = \frac{q+1}{2}$$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2 \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta$$

Express $\int_0^1 \frac{x}{\sqrt{1-x^5}} dx$ in terms of Beta function

Sol:-

$$\text{put } x^5 = y \Rightarrow x = y^{\frac{1}{5}}$$

$$dx = \frac{1}{5} y^{-\frac{4}{5}} dy$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^5}} dx = \int_0^1 \frac{y^{\frac{1}{5}} y^{-\frac{4}{5}}}{\sqrt{1-y}} \frac{dy}{5}$$

$$= \frac{1}{5} \int_0^1 y^{-\frac{3}{5}} (1-y)^{-\frac{1}{2}} dy$$

$$= \frac{1}{5} \beta\left(-\frac{3}{5}+1, -\frac{1}{2}+1\right)$$

$$= \frac{1}{5} \beta\left(\frac{2}{5}, \frac{1}{2}\right)$$

Evaluate

$$\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{1}{a^n b^m} \beta(m, n)$$

$$\text{put } bx = ay \Rightarrow x = \frac{a}{b} y$$

$$dx = \frac{a}{b} dy$$

$$\therefore \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{a^{m-1}}{b^{m-1}} \cdot \frac{a}{b} \int_0^{\infty} \frac{y^{m-1}}{(a+ay)^{m+n}} dy$$

$$= \frac{a^m}{b^m a^{m+n}} \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy = \frac{1}{a^n b^m} \beta(m, n)$$

2) Gamma function: is the Eulerian integral of 2nd kind
denoted by $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$; $n > 0$

remarks $\Gamma(1) = \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty}$
 $= -0 + \frac{1}{e^0} = 1$

$\therefore \boxed{\Gamma(1) = 1}$

Now $\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx$

$$= -x^n e^{-x} \Big|_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-x} dx$$

$$\stackrel{\text{I.I.}}{\downarrow} + n \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= n \Gamma(n)$$

$\therefore \boxed{\Gamma(n+1) = n \Gamma(n)}$ $\rightarrow$ called recurrence formula for $\Gamma(n)$

or $\Gamma(n) = (n-1) \Gamma(n-1)$

Suppose 'n' is a true integer then by above formula

$$\Gamma(n+1) = n \Gamma(n) = n(n-1) \Gamma(n-1) =$$

$$= n(n-1)(n-2) \Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3) \Gamma(n-3)$$

$$= n(n-1)(n-2)(n-3)(n-4) \dots 1 \cdot 3 \cdot 2 \cdot 1 \Gamma(1)$$

$$= n! \Gamma(1) \quad \therefore \boxed{\Gamma(n+1) = n \Gamma(n) = n!}$$

using above for any value of n

$$n\Gamma(n) = \Gamma(n+1)$$

$$\text{or } \boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$$

$\Rightarrow$ for Integral value

$$\boxed{\Gamma(n+1) = n!}$$
$$\dots \text{or } \boxed{\Gamma(n) = (n-1)!}$$

$$\Gamma(1) = 0! = 1$$

$$\therefore \Gamma(2) = 1! = 1$$

$$\Gamma(3) = 2! = 2$$

$$\Gamma(4) = 3! = 6$$

and so on

more over

$$\boxed{\Gamma(0) = \infty}$$

as

$$n\Gamma(n) = \Gamma(n+1)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

put $n=0$

$$\Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

so we have

$$\Gamma(0) = \infty$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n!$$



$n\Gamma(n) = \Gamma(n+1) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$ is also applicable
for non-integers

Transformation of Γ (Gamma) function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

(A)

put $x = ay$

$$\begin{aligned}\Gamma(n) &= \int_0^{\infty} e^{-ay} (ay)^{n-1} a dy \\ &= a^n \int_0^{\infty} e^{-ay} y^{n-1} dy \\ &= a^n \int_0^{\infty} e^{-ay} y^{n-1} dy\end{aligned}$$

$$\text{or } \int_0^{\infty} e^{-ay} y^{n-1} dy = \frac{\Gamma(n)}{a^n}$$

(B)

put $e^{-x} = y$ i.e. $x = \log \frac{1}{y} \Rightarrow dx = -\frac{dy}{y}$

$$\Gamma(n) = \int_1^0 y (\log \frac{1}{y})^{n-1} (-\frac{1}{y}) dy$$

$$\Gamma(n) = \int_0^1 (\log \frac{1}{y})^{n-1} dy$$

 $n > 0$

(C)

put $x^{\frac{1}{n}} = y \Rightarrow x = y^n \quad \& \quad dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

$$\Gamma(n) = \int_0^{\infty} e^{-y^n} (y^{\frac{1}{n}})^{n-1} \cdot \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$\Gamma(n) = \frac{1}{n} \int_0^{\infty} e^{-y^n} dy$$

put $n = \frac{1}{2}$

$$\Gamma(\frac{1}{2}) = \frac{1}{\frac{1}{2}} \int_0^{\infty} e^{-y^2} dy$$

$$= 2 \cdot \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

1) $e^{-x} \rightarrow y$

2) $x \rightarrow ay$

3) $x^n \rightarrow y$

Ques:- Show that

$$\int_0^1 y^{m-1} (\log \frac{1}{y})^{n-1} dy = \frac{\Gamma(n)}{m^n} ; m > 0, n > 0$$

Sol:-

$$\text{put } \log \frac{1}{y} = x$$

$$\frac{1}{y} = e^x$$

$$y = e^{-x}$$

$$dy = -e^{-x} dx$$

$$\text{when } y \neq 0$$

$$x = \log \frac{1}{0} = \log \infty = \infty$$

$$y = 1$$

$$x = \log \frac{1}{1} = 0$$

$$\therefore \int_0^1 y^{m-1} (\log \frac{1}{y})^{n-1} dy = \int_{\infty}^0 e^{-x(m-1)} x^{n-1} (-e^{-x} dx)$$

$$= \int_0^{\infty} e^{-mx} x^{n-1} dx$$

$$\text{put } mx = t \\ dx = \frac{dt}{m}$$

$$= \int_0^{\infty} e^{-t} \left(\frac{t}{m}\right)^{n-1} \frac{dt}{m}$$

$$= \frac{1}{m^n} \int_0^{\infty} e^{-t} t^{n-1} dt$$

$$= \frac{\Gamma(n)}{m^n}$$

$$\therefore \int_0^1 y^{m-1} (\log \frac{1}{y})^{n-1} dy = \frac{\Gamma(n)}{m^n}$$

Hence proved.

Evaluate

$$I = \int_0^{\infty} x^{n-1} e^{-h^2 x^2} dx$$

Soln:-

$$I = \int_0^{\infty} x^{n-1} e^{-h^2 x^2} dx$$

put $y = h^2 x^2 \Rightarrow x = \frac{\sqrt{y}}{h}$
 $dy = 2h^2 x dx$

$$\therefore I = \int_0^{\infty} \left(\frac{y^{\frac{1}{2}}}{h}\right)^{n-1} e^{-y} \frac{dy}{2h^2 \sqrt{y}}$$

$$I = \int_0^{\infty} \frac{y^{\frac{1}{2}(n-1)}}{2h^n} e^{-y} dy = \frac{1}{2h^n} \int_0^{\infty} y^{\frac{n}{2}-1} e^{-y} dy$$

$$I = \frac{1}{2h^n} \Gamma\left(\frac{n}{2}\right)$$

Ans.

show that $\Gamma_n = 2 \int_0^{\infty} e^{-t^2} t^{2n-1} dt$

Proof:- $\Gamma_n = \int_0^{\infty} e^{-x} x^{n-1} dx$

put $x = t^2$
 $dx = 2t dt$

$$\therefore \Gamma_n = \int_0^{\infty} e^{-t^2} t^{2(n-1)} 2t dt$$

$$\Gamma_n = 2 \int_0^{\infty} e^{-t^2} t^{2n-1} dt$$

Hence proved

Relation b/w Beta & Gamma function

we have $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

put $x = \alpha y$
 $dx = \alpha dy$

$$\therefore \Gamma(n) = \int_0^{\infty} e^{-\alpha y} (\alpha y)^{n-1} \alpha dy = \int_0^{\infty} \alpha^n e^{-\alpha y} y^{n-1} dy \quad \text{--- (1)}$$

multiplying b.s. by $e^{-\alpha} \alpha^{m-1}$ & integrate w.r.t α b/w the limits 0 to ∞

$$\Gamma(n) \int_0^{\infty} e^{-\alpha} \alpha^{m-1} d\alpha = \int_0^{\infty} \left[\int_0^{\infty} e^{-\alpha(y+1)} \alpha^{m+n-1} d\alpha \right] y^{n-1} dy$$

$$\Gamma(m) \Gamma(n) = \int_0^{\infty} \frac{\Gamma(m+n)}{(1+y)^{m+n}} y^{n-1} dy \quad \text{using (1)}$$

$$\Gamma(m) \Gamma(n) = \Gamma(m+n) \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy = \beta(m, n)$$

i.e

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Hence proved.

$$* \frac{\Gamma(n)}{\alpha^n} = \int_0^{\infty} e^{-\alpha y} y^{n-1} dy$$