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<u>Defn</u> Let L be a lattice with 0. An element $a \in L$ is called an atom if $a \neq 0$ & if for some $b \in L$, $0 < b \leq a$, then $a = b$.

<u>ex</u> $(\mathbb{N}, \leq)$ has only one atom i.e 2

<u>ex</u> $(\mathbb{N}, '|')$ has infinite no. of atoms. All prime numbers are atoms.

<u>Defn</u> $a \in L$ is called join irreducible if $\forall b, c \in L$, $a = b \vee c$
$\Rightarrow a = b$ or $a = c$.

<u>Lemma</u> Let L be a lattice & $a \in L$ be an atom in L. Then a is join irreducible.
(Prove).

<u>Lemma 2</u> Let L be a distributive lattice & $p \in L$ be join irreducible elt s.t $p \leq a \vee b$ then $p \leq a$ or $p \leq b$.

<u>Pf</u> $p \leq a \vee b$

$\Rightarrow p = p \wedge (a \vee b)$

$= (p \wedge a) \vee (p \wedge b)$.

& p is join irreducible

$\Rightarrow p = p \wedge a$ or $p = p \wedge b$

$\Rightarrow p \leq a$ or $p \leq b$.

<u>Defn</u> Let L be a lattice & $[x, y]$ be an interval in L.
Let $a \in [x, y]$. An elt. $b \in L$ is called relative complement of a w.r.t. $[x, y]$ if $a \wedge b = x$ & $a \vee b = y$

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Prove $b \in [x, y]$.

Defn If each $a \in [x, y]$ has a relative complement w.r.t $[x, y]$, then $[x, y]$ is called complemented.
If each interval $[x, y]$ in L is complemented then L is called relatively complemented.

Defn If L has a zero elt 0 & if each interval $[0, b]$ in L is complemented then L is called sectionally complemented lattice.

ex $(\mathbb{N}, |)$ is not complemented $\because$ it does not have a greatest element.

Remark S = any set
D = Distributive lattice
$L = \{ f \mid f : S \to D \}$
Define $f \leq g$ in L as
$f \leq g$ iff $f(x) \leq_{\text{in } D} g(x) \ \forall x \in S$

Prove $\leq$ is a partial order relation.

Let $f, g \in L$,
Define $f \wedge g : S \to D$ as
$(f \wedge g)(x) = f(x) \wedge g(x) \ \forall x \in S$
& $f \vee g : S \to D$ as
$(f \vee g)(x) = f(x) \vee g(x) \ \forall x \in S$.

Prove ~~$f \wedge g$~~ $\wedge$ & $\vee$ are Commutative associative & absorption holds in L.

$\therefore (L, \leq, \wedge, \vee)$ is a lattice.

Prove Distributivity

$\therefore$ L is a dist.. lattice.

Remark let $(P, \leq)$ be a poset
Define a relation $(<)$ on P as
$x < y$ in P iff $x \leq y$ & $x \neq y$.

Q1.) let P be a set on which a binary relation $(<)$ is defined s.t
$\forall x, y, z \in P$ a) $x < x$ is false
& b) If $x < y$ & $y < z$ then $x < z$
Prove that if $(\leq)$ is defined on P as $x \leq y$ iff $(x < y$ or $x = y)$
Then $\leq$ is a Partial order relation on P.

Pf Define $\leq$ on P as
$x \leq y$ iff $(x < y$ or $x = y)$

Reflexive $x = x \ \forall x \in P$
$\Rightarrow x \leq x \ \forall x \in P$

Antisymm: let $x \leq y$ & $y \leq x$
$\Rightarrow (x < y$ or $x = y)$ & $(y < x$ or $y = x)$

Case 1 $x < y$ & $y < x \Rightarrow x < x$ by (b) false

Case 2 $x < y$ & $y = x \Rightarrow x < x$ (by a) false

Case 3 $x = y$ & $y < x \Rightarrow x < x$ (by a) false.

Case 4 $x = y$ & $y = x \Rightarrow x = y$ (True)

$\therefore \leq$ is antisymmetric.

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Transitive let $x \le y$ & $y \le z$
$\Rightarrow (x < y$ or $x = y)$ & $(y < z$ or $y = z)$

Case 1 $x < y$ & $y < z \Rightarrow x < z$ (by (b))
$\Rightarrow x \le z$

Case 2 $x < y$ & $y = z \Rightarrow x < z$ ~~(by~~
$\Rightarrow x \le z$

Case 3 $x = y$ & $y < z \Rightarrow x < z$
$\Rightarrow x \le z$

Case 4 $x = y$ & $y = z \Rightarrow x = z \Rightarrow x \le z$

$\therefore (P, \le)$ is a Poset.

2nd Part of Q 1.1 (Converse)
Every order relation $\le$ on P arises from a relation '$<$' on P satisfying (a) & (b).

If $\le$ is a Partial order rel. on P.

Define $<$ on P as $x < y$ in P Iff $(x \le y$ & $x \ne y)$

(a) let $x < x$ be true
$\Rightarrow x \le x$ & $x \ne x$
Which is absurd
$\therefore x < x$ is false. the

(b) let $x < y$ & $y < z$
$\Rightarrow (x \le y$ & $x \ne y)$ & $(y \le z$ & $y \ne z)$
$x \le y$ & $y \le z \Rightarrow x \le z$ (By transi.)
Suppose $x = z$
Then $z \le y$ & $y \le z \Rightarrow y = z$
Contradiction
$\therefore x \ne z \Rightarrow x < z$
$\therefore$ (a) & (b) Proved.