Equations of the first Order, But not of first degree :-

The equation of the first order and nth degree is

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1} p + a_n = 0 \quad (1)$$

where $p = \frac{dy}{dx}$ and $a_1, a_2, \ldots, a_n$ are functions of x and y.

To find solution of eqn (1) we will discuss following Cases :-

Case (I) :- If equation (1) that can be resolved into Component equations of the first degree, or if equation (1) can take the form

$$(p-R_1)(p-R_2)\cdots(p-R_n)=0 \quad (2)$$

Eqn (1) is satisfied by a value of y that will make any factor of the eqn (2) equal to zero. Therefore, to obtain the soln of (1), equate each of the factors in eqn (2) to zero, and obtain the soln of the n equations. The n solutions can be left distinct or Combined into one.

Suppose soln derived for (2) are

$f_1(x,y,c_1)=0,\ f_2(x,y,c_2)=0, \ldots, f_n(x,y,c_n)=0$, where $c_1, c_2, \ldots c_n$ are the arbitrary constants of integration.

Then solns are evidently just as general, if

$c_1 = c_2 = \dots = c_n$. Since all the c's can have any one of an infinite number of values: and the solns will be

$$f_1(x,y,c)=0,\ f_2(x,y,c)=0,\ \dots,\ f_n(x,y,c)=0.$$

These can be combined into one eqn,

$$f_1(x,y,c)\,f_2(x,y,c)\dots f_n(x,y,c)=0 \quad \text{--- (3)}$$

eqn (3) is a general soln of eqn (1).

Example:- (1) $p^3+2xp^2-y^2p^2-2xy^2p=0$ — (1) can be written as

$$p(p+2x)(p-y^2)=0.$$

its component eqns are

$$p=0,\quad p+2x=0,\quad p-y^2=0$$

or

$$\frac{dy}{dx}=0,\quad \frac{dy}{dx}=-2x,\quad \frac{dy}{dx}=y^2$$

solns are

$$y=c,\quad y+x^2=c,\quad xy+cy+1=0$$

Then, the soln of (1) is (Combined soln)

$$\boxed{(y-c)(y+x^2-c)(xy+cy+1)=0}$$

Example 2:- $\left(\frac{dy}{dx}\right)^2 - ax^3 = 0$ ——(1)

then

$$\frac{dy}{dx} = \pm a^{1/2} x^{3/2} \quad \text{——(2)}$$

By Integrating (2), we get

$$y + c = \pm \frac{2}{5} a^{1/2} x^{5/2}$$

$$(y+c)^2 = \frac{4}{25} a x^5$$

$$\boxed{25(y+c)^2 - 4ax^5 = 0} \text{ Ans}$$

Exercise:- (1) Solve $p^3(x+2y) + 3p^2(x+y) + (y+2x)p = 0$.

(2) $\left(\frac{dy}{dx}\right)^3 = ax^4$.

(3) $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$.

(4) $p^2 - 7p + 12 = 0$.

Case (II):- Equations ~~of~~ solvable for y:-

If the equation $f(x, y, p) = 0$, can not be resolved into component equations and can be put in the ~~form~~ form $y = F(x, p)$. ——(1)

differentiation with respect to x gives

$p = \phi\left(x, p, \frac{dp}{dx}\right)$, —(2) which is an equation in two variables x and p. By solving eq^n (2)

We can find relation

$$\psi(x, p, c) = 0 \quad \text{--- (3)}$$

Now, eliminate 'p' between the latter (3) and the original equation (1), we can find solⁿ.

When elimination of p between eqⁿ (1) & (3) is not easily possible, the values of x and y in terms of p as a parameter can be found, and them together will constitute the solⁿ.

Example :- $x - yp = ap^2 \quad \text{--- (1)}$

Here $y = \dfrac{x - ap^2}{p} \quad \text{--- (2)}$

Differentiating (2) with respect to x

$$\frac{dy}{dx} = \frac{p\left(1 - a \times 2p\frac{dp}{dx}\right) - (x - ap^2)\frac{dp}{dx}}{p^2}$$

$$p = \frac{p - 2ap^2\frac{dp}{dx} - (x - ap^2)\frac{dp}{dx}}{p^2}$$

$$p^3 = p - (x + ap^2)\frac{dp}{dx}$$

or

$$(x + ap^2)\frac{dp}{dx} = p - p^3$$

or

$$(x + ap^2)\frac{dp}{dx} = p(1 - p^2) \quad \text{--- (3)}$$

or

$$\boxed{\frac{dx}{dp} - \frac{1}{p(1-p^2)}x = \frac{ap}{1-p^2}} \quad \text{--- (4)}$$

eqⁿ (4) is in the form $\boxed{\dfrac{dy}{dx} + f(x)\, y = g(x)}$ which is Linear differential eqⁿ of 1st order.

~~Solⁿ of~~ I.F. of L.D.E. is $e^{\int f(x)dx}$

and solⁿ of L.D.E. is

$$\boxed{y \times e^{\int f(x)dx} = \int \left[e^{\int f(x)dx} \times g(x) \right] dx + c}$$

Now I.F. of eqⁿ (4) is $e^{-\int \frac{1}{p(1-p^2)} dp}$

$$= e^{-\int \left[\frac{1}{p} - \frac{1}{2(1+p)} + \frac{1}{2(1-p)} \right] dp}$$

$$= e^{-\left[\log p - \frac{1}{2}\log(1+p) + \frac{1}{2}\log(1-p) \right]}$$

$$= e^{-\left[\log p - \frac{1}{2}\log(1-p^2) \right]}$$

$$= e^{-\left[\log p - \log(1-p^2)^{1/2} \right]}$$

$$\text{I.F.} = e^{-\log \frac{p}{(1-p^2)^{1/2}}} = \frac{(1-p^2)^{1/2}}{p}$$

then solⁿ of (4) is

$$x \times \frac{(1-p^2)^{1/2}}{p} = \int \frac{ap}{(1-p^2)} \times \frac{(1-p^2)^{1/2}}{p} dp + c$$

$$\frac{x(1-p^2)^{1/2}}{p} = \int \frac{a}{(1-p^2)^{1/2}} dp + c$$

$$\frac{x(1-p^2)^{1/2}}{p} = a \sin^{-1} p + c$$

$$\boxed{x = \frac{p}{(1-p^2)^{1/2}} \left[a \sin^{-1} p + c \right]} \quad \text{--- (5)}$$

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Putting the value of x in eqn (2), we get

$$y = -ap + \frac{1}{(1-p^2)^{1/2}}(a \sin^{-1} p + c) \quad \text{--- (6)}$$

eqn (5) & (6) together give the soln of eqn (1),

Exercise:- (1) Solve $y = x + a \tan^{-1} p$.

(2) Solve $4y = x^2 + p^2$.

(3) Solve $xp^2 - 2yp + ax = 0$,

Case III:- Equations solvable for x:-

When it is difficult to resolved eqn $f(x, y, p) = 0$ — (1) into components eqn's but can be put in the form

$$x = F(y, p) \quad \text{--- (2)}$$

Differentiating of (2) with respect to y gives.

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \quad \text{--- (3)}$$

after solving eqn (3) ~~Now, eliminating p from eqn (1) & (3) we got~~ we can get

$$\psi(y, p, c) = 0 \quad \text{--- (4)}$$

Now eliminate p from eqn (1) & (4) we can get required soln $f_1(x, y, c) = 0$ of (1).

If elimination of p is not possible then x and y expressed in term of p as in case (II) will give required soln, Example :- $x = y + p^2$ —— (1)

Differentiation of (1) with respect to y, gives

$$\frac{1}{p} = \frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

or $$2p\frac{dp}{dy} = \frac{1}{p} - 1$$

$$\frac{dp}{dy} = \frac{1}{2p^2} - \frac{1}{2p}$$

or $$dy = \frac{2p^2}{1-p}\,dp$$

$$dy = -\left[2(p+1) + \frac{2}{p-1}\right]dp$$

By Integrating $$y = -\left[2\left(\frac{p^2}{2} + p\right) - 2\log(p-1)\right] + c$$

$$y = -p^2 - 2p + 2\log(p-1) + c$$

$$\boxed{y = -p(p-2) + 2\log(p-1) + c} \quad —— (2)$$

putting value of y in (1) we get

$$\boxed{x = -2p + 2\log(p-1) + c} \quad —— (3)$$

eqn (2) & (3) together gives soln of (1).

Exercise :- (1) Solve $x = y + a\log p$

(2) Solve $p^2y + 2px = y$