with x-axis. Its projection ON_2 on x-axis is second harmonic motion of same frequency ω, amplitude a_2 and phase constant α_2. The superposition of two motions is given by the vector OP which can be obtained by parallelogram law of vector sum of OP_1 and P_1P as shown in Fig (1.6b), the vector P_1P is equal to vector OP_2. Since $\underline{|P_1ON_1} = \omega t + \alpha_1$ and $\underline{|P_2ON_2} = \omega t + \alpha_2$, the angle between OP_1 and P_1P is just $\alpha_2 - \alpha_1 = \alpha$. Thus, we have from rt. angle triangle POQ,

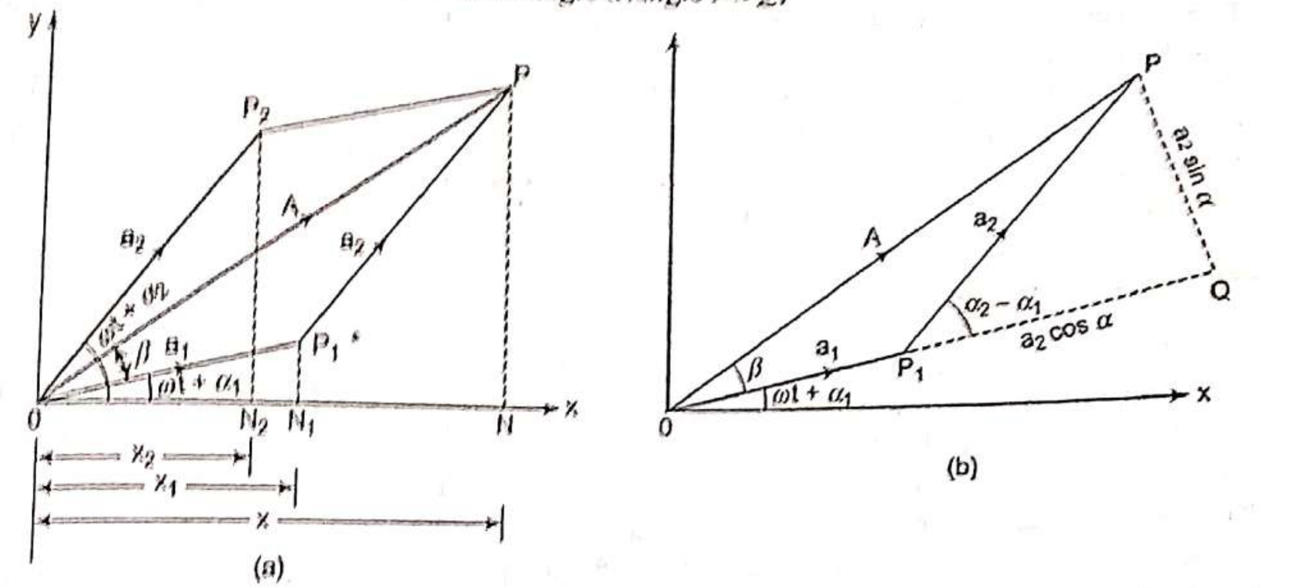

Fig. 1.6. *Addition of vectors, each representing simple harmonic motion along x-axis at angular frequency ω to give a resulting SHM displacement*

$$A^2 = (a_1 + a_2 \cos \alpha)^2 + (a_2 \sin \alpha)^2$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \alpha$$

which is same as obtained analytically (eqn. 1.36)

The total phase of resultant motion is given by $\underline{|PON}$ and let this is equal to $(\omega t + \delta)$ where δ is the phase constant of the resultant motion. From Fig. (1.6b), we have

$$\delta = \beta + \alpha_1$$

$\therefore$

$$\tan \delta = \tan (\beta + \alpha_1) = \frac{\tan \beta + \tan \alpha_1}{1 + \tan \beta \tan \alpha_1}$$

now

$$\tan \beta = \frac{a_2 \sin \alpha}{a_1 + a_2 \cos \alpha} \qquad \text{(where } \alpha = \alpha_2 - \alpha_1\text{)}$$

Putting for $\tan \beta$ in the above equation and simplifying, we get

$$\tan \delta = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}$$

which is same as eqn. (1.37) obtained analytically.

1.5 SUPERPOSITION OF TWO SIMPLE HARMONIC OSCILLATIONS OF DIFFERENT FREQUENCIES: BEATS

When two simple harmonic waves of slightly differing frequencies (*e.g.*, from two tuning forks of nearly equal frequencies) travel along the same straight line in the same direction, then the resultant amplitude is alternately maximum and minimum. Thus the intensity of sound, which is proportional to the square of amplitude, rises and falls (technically known as waxing and waning of sound) alternately with time. This phenomenon of waxing and waning of sound is called beats. One waxing and one waning constitutes one beat. The number of waxing and waning in one second is called the frequency of beats. This frequency of beats is equal to the difference in the frequencies of the sound waves.

Production of beats (Graphical). The phenomenon of beats occurs as a result of superposition of two sound waves of slightly different frequencies travelling along the same straight line in the same direction. Consider that a particular instant (Fig. 1.7), the two waves meet in the same phase at a particular point. They reinforce to produce maximum sound intensity. After this instant, they get further and further out of phase as their frequencies are slightly different. After a short time (at time t_2) the two waves arrive at the point in the opposite phase. This happens when one wave gains half a vibration on the other. Now they produce minimum sound intensity. Again after some time *i.e.*, at instant t_3 one wave gains one full vibration on the other and the two waves are again in phase and produce maximum sound intensity, and so on. One maximum and one minimum constitute one beat. The number of beats per sec. is equal to the difference in frequencies of the sources.

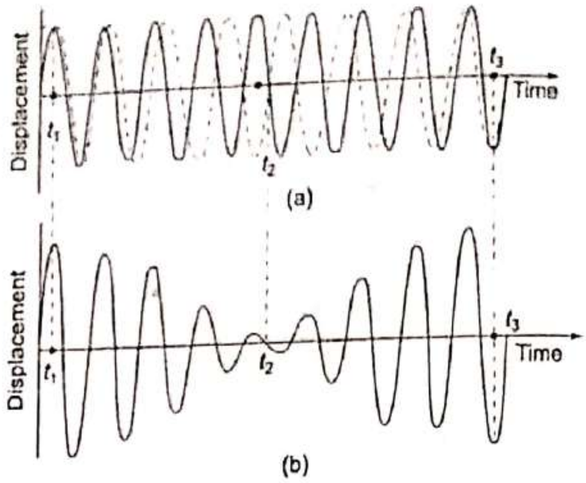

Fig. 1.7 *(a) Two harmonic oscillations of slightly different frequencies. (b) Resultant oscillation due to superposition.*

Now we shall explain the production of beats by considering the case of two tuning forks of frequencies 256 and 254. Let the two forks start vibrating together in the same phase. After 1/4 second, the first fork completes its 64 vibrations while the second one has completed its 63½ vibrations. The two waves are now in opposite phase and produce minimum intensity. After 1/2 second, the two waves are again in phase (phase difference is equal to λ) and produce maximum intensity. After 3/4 second, the first fork completes 192 vibrations while the second one completes 190½ vibrations. There is phase change of $3\lambda/2$ *i.e.*, the two waves are in opposite phase and produce minimum intensity. After completion of one second, they are again in phase and produce maximum sound intensity. During one sec. two maxima and two minima are recorded *i.e.*, two beats are heard in one sec. Hence the number of beats is equal to the difference in the frequencies of the two sources.

Mathematical analysis. Consider the case of two waves having same amplitude a with slightly different frequencies n_1 and n_2 traveling simultaneously in medium. If y_1 and y_2 be the displacements of these waves at any instant t, then

$$y_1 = a \sin 2\pi n_1 t \qquad \text{...(1.40)}$$

and

$$y_2 = a \sin 2\pi n_2 t \qquad \text{...(1.41)}$$

Applying the principle of superposition, the resultant displacement y at any instant t is given by,

$$y = y_1 + y_2 = a \sin 2\pi n_1 t + a \sin 2\pi n_2 t$$

$$= 2a \sin 2\pi\left(\frac{n_1+n_2}{2}\right)t \,.\, \cos 2\pi\left(\frac{n_1-n_2}{2}\right)t = \left[2a \cos 2\pi\left(\frac{n_1-n_2}{2}\right)t\right] \times \sin 2\pi\left(\frac{n_1+n_2}{2}\right)t$$

or

$$y = A \sin 2\pi\left(\frac{n_1+n_2}{2}\right)t \qquad \text{...(1.42)}$$

where $A = 2a\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t$. Equation (1.42) represents a simple harmonic motion whose amplitude is A and average frequency $= (n_1 + n_2)/2$.

Now for A to be maximum, $\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t = \pm 1$

or $$2\pi\left(\frac{n_1 - n_2}{2}\right)t = k\pi \text{ where } k = 0, 1, 2, \ldots$$

or $$t = \frac{k}{(n_1 - n_2)}$$

when $k = 0$, $t = 0$ First maxima

$k = 1$, $t_1 = \frac{1}{(n_1 - n_2)}$ Second maxima

$k = 2$, $t_2 = \frac{2}{(n_1 - n_2)}$ Third maxima

....

$k = n$, $t_n = \frac{n}{(n_1 - n_2)}$ nth maxima

Thus the time interval between two successive maxima $= \frac{1}{(n_1 - n_2)}$ seconds.

$\therefore$ Frequency of maxima $= (n_1 - n_2)$.

Similarly, the amplitude is minimum, when

$$\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t = 0$$

or $$2\pi\left(\frac{n_1 - n_2}{2}\right)t = (2k + 1)\pi/2, \quad \text{where } k = 0, 1, 2, 3, \ldots$$

or $$t = \frac{(2k + 1)}{2(n_1 - n_2)}$$

where $k = 0$, $t = \frac{1}{2(n_1 - n_2)}$

$k = 1$, $t_1 = \frac{3}{2(n_1 - n_2)}$

$k = 2$, $t_2 = \frac{5}{2(n_1 - n_2)}$

....

$k = n$, $t_n = \frac{(n + 1)}{2(n_1 - n_2)}$

The interval between two successive minima $= \frac{1}{(n_1 - n_2)}$ sec

Frequency of minima $= (n_1 - n_2)$.

This shows that the frequency of minima is the same as that of maxima. Between any two maxima, there is a minimum. *The periodic variation of the amplitude of the motion, resulting from superposition of SHM's of slightly different frequencies, is known as the phenomenon of beats.* One maximum of amplitude followed by a minimum is technically called *beat.* In one sec. the intensity is maximum $(n_1 - n_2)$ time and minimum also $(n_1 - n_2)$ times. Hence $(n_1 - n_2)$ beats will be heard in one second. Hence the beat frequency is equal to the difference between the frequencies of the component oscillations.

Figure 1.7 shows graphically the result of superposing two harmonic oscillations of slightly different frequencies. It may be noticed that two oscillations in Fig 1.7(a) are harmonic while their superposition shown in Fig. 1.7(b) *is periodic but not harmonic.*

There are number of application of beats. Phenomenon of beats can be used to determine the small difference between frequencies of two sources of sound. Musicians often use this phenomenon in tuning their instruments. Sometimes beats are deliberately produced in a particular section of an orchestra to give a pleasing tone to the resulting sound. The phenomenon of beats is also used to transmit a signal from one place to another. The beats called wave groups or wave packets propagate in space.

1.6 SUPERPOSITION OF *N* HARMONIC OSCILLATIONS

In previous sections we have discussed the superposition of two harmonic oscillations and the method can be extended to any large number of oscillations in which the frequencies, amplitudes and initial phases of the component oscillations are all different. We shall discuss the following two cases:

(i) Superposition of n harmonic oscillations, all having same frequency and amplitude but with equal successive initial phase differences.

(ii) Superposition of n harmonic oscillations, all having same amplitude and initial phase difference but with equal successive frequency differences.

1.6.1 Superposition of n Harmonic Oscillations with Equal Phase Differences

Let us consider the superposition of n harmonic oscillations each of amplitude a, angular frequency ω and the initial phase differing from its neighbouring oscillation by an angle δ. The first these component oscillations can be described by the equation

$$x_1 = a \cos \omega t$$

and the other oscillations are thus given by

$$x_2 = a \cos (\omega t + \delta)$$

$$x_2 = a \cos (\omega t + 2\delta)$$

$$\cdots\cdots\cdots\cdots$$

$$x_n = a \cos \{\omega t + (n-1)\delta\}$$

According to superposition principle, the resultant motion can be given as

$$x = a \cos \omega t + a \cos(\omega t + \delta) + a \cos (\omega t + 2\delta) + \ldots + a \cos \{\omega t + (n-1)\delta\} \quad (1.43)$$

This resultant motion can be obtained by the following graphical method.

Graphical Method: The graphical representation of the expression (1.43) is shown in Fig. (1.8). Vectors $AA_1, A_1A_2 \ldots A_{n-1}A_n$ respectively represent the first, second … and the nth harmonic oscillation. Vector AA_n represents the resultant vector and its length R is the resultant amplitude. The combining vectors form successive sides of a regular polygon. Any regular polygon can be inscribed in a circle having same radius with its centre at a point O. All the corners $A, A_1, A_2 \ldots A_n$ lie on the circle and the angle subtended at O by any individual vector e.g. AA_1, is equal to the angle δ between adjacent vectors. Hence the total angle AOA_n subtended at O by the resultant vector is equal to $n\delta$. It is evident from the Fig. (1.8) that the projection of AA_n on the x-axis gives the resultant displacement i.e.

$$x = R\cos(\omega t + \alpha)$$

where α is the angle between x-axis and AA_1 (not shown in Fig) and R is the amplitude of the resultant, given as

$$R = 2r\sin\frac{n\delta}{2} = a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase with respect to the first contribution is given by

$$\alpha = (n-1)\delta/2$$

where α is its phase difference with respect to the first component $a\cos\omega t$.

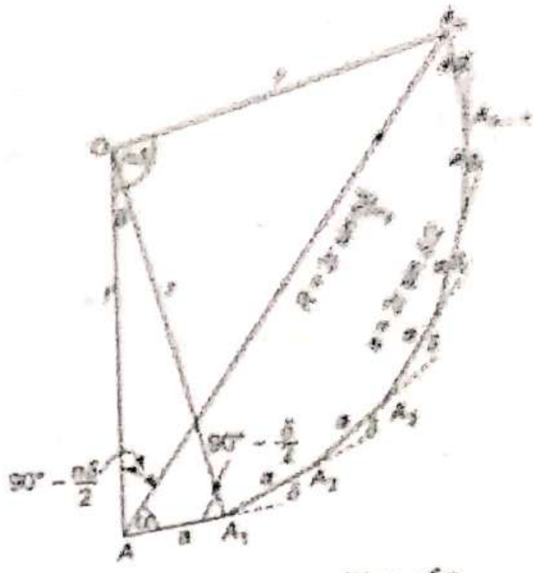

Fig. 1.8 *Superposition of n harmonic oscillations of equal amplitude a*

Geometrically we see that each length

$$a = 2r\sin\frac{\delta}{2}$$

where r is the radius of the circle enclosing the (incomplete) polygon. From the isosceles triangle OAC the magnitude of the resultant

$$R = 2r\sin\frac{n\delta}{2} = a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase angle is seen to be

$$\alpha = \angle OAA_1 - \angle OAA_n$$

In the isosceles triangle OAA_n

$$\angle OAA_n = 90° - \frac{n\delta}{2}$$

and in the isosceles triangle OAA_1

$$\angle OAA_1 = 90° - \frac{\delta}{2}$$

so

$$\alpha = \left(90° - \frac{\delta}{2}\right) - \left(90° - \frac{n\delta}{2}\right) = (n-1)\frac{\delta}{2}$$

that is, half the phase difference between the first and the last contributions. Hence the resultant

$$x = R\cos(\omega t + \alpha) = a\frac{\sin n\delta/2}{\sin\delta/2}\cos\left[\omega t + (n-1)\frac{\delta}{2}\right]$$

For the moment let us examine the behavior of the magnitude of the resultant

$$R = a\frac{\sin n\delta/2}{\sin\delta/2}$$

which is not constant but depends on the value of δ. When n is very large δ is very small and the polygon becomes an arc of the circle centre O, of length $na = l$ with R as the chord. Then

$$\alpha = (n-1)\frac{\delta}{2} \approx \frac{n\delta}{2}$$

and

$$\sin\frac{\delta}{2} \to \frac{\delta}{2} \approx \frac{\alpha}{n}$$

Hence, in this limit,

$$R = a\frac{\sin n\delta/2}{\sin\delta/2} = a\frac{\sin\alpha}{\alpha/n} = na\frac{\sin\alpha}{\alpha} = \frac{l\sin\alpha}{\alpha}$$

The behaviour of $l \sin\alpha/\alpha$ versus α is shown in Fig. 1.9. The pattern is symmetric about the value $\alpha = 0$ and is zero whenever $\sin\alpha = 0$ except at $\alpha \to 0$ that is, when $\sin\alpha/\alpha \to 1$. When $\alpha = 0$, $\delta = 0$ and the

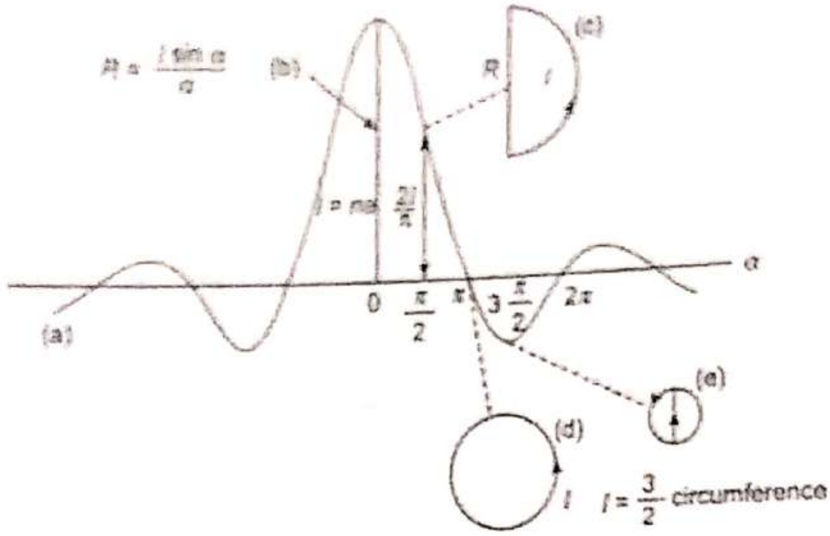

Fig. 1.9. *(a) Graph of $l \sin\alpha/\alpha$ versus α, showing the magnitude of the resultants for (b) $\alpha = 0$; (c) $\alpha = \pi/2$; (d) $\alpha = \pi$ and (e) $\alpha = 3\pi/2$*

resultant of the n vector is the straight line of length l, Fig. 1.9(b). As δ increases l becomes the arc of a circle until at $\alpha = \pi/2$ the first and last contributions are out of phase ($2\alpha = \pi$) and the arc l has become a semicircle of which the diameter is the resultant R Fig. 1.9(c). A further increase in δ increases α and curls the constant length l into the circumference of a circle ($\alpha = \pi$) with a zero resultant, Fig. 1.9(d). At $\alpha = 3\pi/2$, Fig. 1.9(e) the length l is now 3/2 times the circumference of a circle whose diameter is the amplitude of the first minimum.

1.6.2 Superposition of n Harmonic Oscillations with Equal Frequency Differences

Consider superposition of n different harmonic oscillations having equal amplitudes a, equal phase constants (assumed to be zero) and angular frequencies distributed uniformly between the lowest frequency ω_1 and the heighest frequency, ω_2. The component oscillations can be described by the equations

$$x_1 = a\cos\omega_1 t$$

$$x_2 = a\cos(\omega_1 + \delta\omega)t$$

$$x_3 = a\cos(\omega_1 + 2\delta\omega)t$$

$$\text{- - - - - - - - - - - - - - - -}$$

$$x_n = a\cos\{\omega_1 t + (n-1)\delta\omega t\} = a\cos\omega_2 t$$

According to superposition principle, the resultant motion can be obtained as

$$x = a\cos\omega_1 t + a\cos(\omega_1 + \delta\omega)t + a\cos(\omega_1 + 2\delta\omega)t + \dots + a\cos\{\omega_1 t + (n-1)\delta\omega t\} \quad (1.44)$$

where $\delta\omega$ is the frequency spacing between the neighbouring components i.e.

$$\delta\omega = \frac{\omega_2 - \omega_1}{n-1} = \frac{\Delta\omega}{n-1} \quad (1.45)$$

where $\Delta\omega = \omega_2 - \omega_1$ is called the bandwidth. The superposition (1.44) is the real part of the complex function $f(t)$ where

$$f(t) = a[e^{i\omega_1 t} + e^{i(\omega_1+\delta\omega)t} + e^{i(\omega_1+2\delta\omega)t} + \dots + e^{i(\omega_1 t+(n-1)\delta\omega)t}]$$

$$= a e^{i\omega_1 t}[1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}]$$

$$= a e^{i\omega_1 t} S$$

where $\alpha = e^{i\delta\omega t}$

and $S = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1}$ (1.46)

Multiply by α gives $\alpha S = \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{n-1} + \alpha^n$ (1.47)

Subtracting eqn. (1.46) from (1.47), we get

$$(\alpha - 1)S = \alpha^n - 1$$

or

$$S = \frac{\alpha^n - 1}{\alpha - 1}$$

$$= \frac{e^{in\delta\omega t} - 1}{e^{i\delta\omega t} - 1}$$

$$= \frac{e^{in\delta\omega t/2}}{e^{i\delta\omega t/2}}\left(\frac{e^{in\delta\omega t/2} - e^{-in\delta\omega t/2}}{e^{i\delta\omega t/2} - e^{-i\delta\omega t/2}}\right)$$

$$= \exp\left[\frac{i(n-1)}{2}\delta\omega t\right]\frac{\sin(n\delta\omega t/2)}{\sin(\delta\omega t/2)}$$

Thus $f(t)$ is given by

$$f(t) = \exp i\left[\omega_1 t + \frac{n-1}{2}\delta\omega\right]\frac{\sin(n\delta\omega t/2)}{\sin(\delta\omega t/2)}$$

From eqn. (1.45), we get

$$\omega_1 + \frac{1}{2}(n-1)\delta\omega = \omega_1 + \frac{1}{2}(\omega_2 - \omega_1) = \frac{1}{2}(\omega_1 + \omega_2) = \omega_a$$

where ω_a is the average of the two extreme frequencies. Thus we have

$$f(t) = a e^{i\omega_a t}\frac{\sin(n\delta\omega t/2)}{\sin(\delta\omega t/2)}$$

The resultant motion x in eqn. (1.44) is the real part of $f(t)$, thus x is given by

$$x = a\frac{\sin(n\delta\omega t/2)}{\sin(\delta\omega/2)}\cos\omega_a t \quad (1.48)$$

or

$$x = a_m \cos\omega_a t \quad (1.49)$$

where a_m, the modulation amplitude, is given by

$$a_m = a_0 \frac{\sin(n\delta\omega t/2)}{\sin(\delta\omega t/2)} \qquad (1.50)$$

Since a_m is time-dependent, the resulting oscillation is not harmonic though periodic. This result is similar to that of beats. The concept is used in problem of propagation of wave groups or packets.

1.7 SUPERPOSITION OF TWO PERPENDICULAR HARMONIC OSCILLATIONS

In physics there are many instances in which two linear simple harmonic oscillations at *right angles* are combined. The resulting motion is the sum of two independent oscillations. It was demonstrated by Lissajous in 1857 that when *a particle is acted upon simultaneously by two simple harmonic motions at right angles to each other, the resultant path traced out by the particle is a curve. The curves thus obtained are called Lissajous figures.*

The nature of the figures depends upon the following factors: *(i)* amplitude of the waves, *(ii)* frequencies of the two waves and *(iii)* the phase difference between the two waves.

Analytical method: We shall discuss the analytical treatment of these figures for two perpendicular oscillations having different values of phase difference and their frequencies in the ratio of 1 : 1 and 2 : 1 respectively.

1.7.1 Two Simple Harmonic Oscillations Perpendicular to Each Other with Same Frequency but Different Amplitudes

Let us consider two simple harmonic motions having the same frequency (or time period), one acting along the x-axis and the other along the y-axis. Let the two vibrations be represented by

$$x = a\cos\omega t \qquad ...(1.51)$$

and

$$y = b\cos(\omega t + \phi) \qquad ...(1.52)$$

where a and b are the amplitudes of x and y vibrations respectively. The y motion is ahead of the y motion by an angle ϕ *i.e.*, the phase difference between two vibrations is ϕ. The equation of resultant vibration can be obtained by eliminating t between equations (1.50) and (1.51), so that we are left an expression involving only x, y and the constant ϕ. From eqn. (1.51), we have

$$\frac{x}{a} = \cos\omega t$$

but

$$\sin\omega t = \sqrt{1-\cos^2\omega t} = \sqrt{1-x^2/a^2}$$

Expanding eqn. (1,52), we get

$$\frac{y}{b} = \cos\omega t\cos\phi - \sin\omega t\sin\phi$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$, we get

$$\frac{y}{b} = \frac{x}{a}\cos\phi - \sin\phi\sqrt{1-x^2/a^2}$$

or

$$\frac{x}{a}\cos\phi - \frac{y}{b} = \sin\phi\sqrt{1-x^2/a^2}$$

squaring both sides, we get

$$\left(\frac{x}{a}\cos\phi - \frac{y}{b}\right)^2 = \sin^2\phi\left(1-\frac{x^2}{a^2}\right)$$

or $$\frac{x^2}{a^2}\cos^2\phi + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi - \frac{x^2}{a^2}\sin^2\phi$$

or $$\frac{x^2}{a^2}(\cos^2\phi + \sin^2\phi) + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi \qquad (1.53)$$

This is the general equation of an oblique ellipse whose axes are inclined to the co-ordinate axes. *Thus the path followed by the particle, which is subjected to two perpendicular simple harmonic motions of equal frequencies, is an ellipse.* Here we consider some important cases.

(i) When $\phi = 0$ (two vibrations are in phase) in this case $\sin\phi = 0$ and $\cos\phi = 1$

The eqn. (1.53) becomes $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

or $$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0 \text{ or } \pm\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

or $$\pm y = \pm \frac{b}{a}x \qquad ...(1.54)$$

This represents two coincident straight lines passing through the origin and inclined to x-axis at the angle θ, given by

$$\theta = \sin^{-1}(b/a)$$

The resultant motion is rectilinear and takes place along a diagonal of a rectangle of sides $2a$ and $2b$ such that x and y always have the same sign, both positive or both negative (Fig. 1.10a). From eqns. (1.51) and (1.52), by setting $\phi = 0$, we get

$$x = a\cos\omega t$$

$$y = b\cos\omega t$$

which gives $$y = \frac{b}{a}x,$$

this is equation of the straight line of slope b/a. At time $t = 0$, we have, $x = a, y = b$, so that the particle is at P at $t = 0$ (Fig 1.10a). As the time passes the cosines begin to decrease until x and y (in eqns. 1.51 and 1.52) become zero when $\omega t = \pi/2$. The particle moves from P to O. After this time, x and y become negative and at time when $\omega t = \pi$, we have $x = -a$ and $y = -b$, the particle moves from O to P'. After this the particle retraces its path. The particle continues to vibrate along the straight line POP'. In optics, it is called *linearly polarized vibrations.*

(ii) When $\phi = \pi/4$, we have,

$$\sin\phi = \frac{1}{\sqrt{2}} \text{ and } \cos\phi = \frac{1}{\sqrt{2}}$$

Now eqn. (1.53) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2} \qquad (1.55)$$

This represents an oblique ellipse, as shown in Fig. 1.10 (b).

(iii) When $\phi = \pi/2$, we have,

$$\sin\phi = 1 \text{ and } \cos\phi = 0$$

The eqn. (1.53) reduces to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1.56)$$

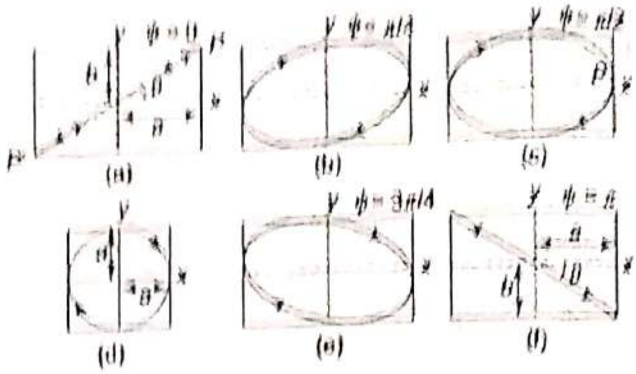

Fig. 1.10 *Lissajous figures, when two perpendicular simple harmonic motions of the same frequency and various phase difference superpose.*

The resultant path is an ellipse whose major and minor axis coincide with the co-ordinate axes as shown in Fig. 1.10 (c). The particle moves in an elliptical path. From eqns. (1.51) and (1.52), at time $t = 0$ and $\phi = \pi/2$, we get $x = a$ and $y = 0$ and the particle is at point P at this time (Fig. 1.10c). As the time t begins to increase from zero, x starts decreasing from its maximum positive value a and y begins to go negative. At time when $\omega t = \pi/2$, $x = 0$ and $y = -b$. The particle moves from P to Q during this time. The subsequent motion of the particle is indicated by arrows in the diagram (Fig 1.10c). The particle traces out an ellipse in the clock wise direction. In optics it is called right handed elliptically polarized vibrations. If $a = b$, then $x^2 + y^2 = a^2$, the resultant path of the particle is a circle of radius a as shown in Fig. 1.10(d).

Thus, *two harmonic oscillations, at right angle to each other, of equal amplitudes and equal frequencies but with phase difference of $\pi/2$, are equivalent to a uniform circular motion, the radius of the circle being equal to the amplitude of either oscillation.*

(iv) When $\phi = 3\pi/4$, we have, $\quad \sin\phi = \dfrac{1}{\sqrt{2}}$ and $\cos\phi = -\dfrac{1}{\sqrt{2}}$

The eqn. (1.53) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\left(-\frac{1}{\sqrt{2}}\right)=\frac{1}{2} \qquad ...(1.57)$$

This represents an oblique ellipse as shown in Fig. 1.10 (e).

(v) When $\phi = \pi$, we have, $\quad \sin\phi = 0$ and $\cos\phi = -1$

Now eqn (1.53) reduces to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}=0$$

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2 = 0 \text{ or } \pm\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$\pm y = \pm\frac{b}{a}x \qquad (1.58)$$

This again represents a pair of coincident straight lines passing through the origin and inclined to x-axis at an angle θ, given by

$$\theta = \tan^{-1}\left(-\frac{b}{a}\right)$$