Ring:

$\underline{R}$ and $\underline{R}$

A ring R is a set with two binary operations usually addition and multiplication which satisfies the following properties:

or

A binary structure $(R, +, \cdot)$ is called ring if it satisfies the following properties:

I $(R, +)$ is an abelian group

(i) Closure Property: $a+b \in R \;\forall\; a, b \in R$

(ii) Associativity $a+(b+c) = (a+b)+c \;\forall\; a, b, c \in R$

(iii) Identity for every $a \in R$, $\exists\; 0 \in R$ such that
$a+0 = a = 0+a$

(iv) Inverse: for every $a \in R$, $\exists\; -a \in R$ such that $a+(-a) = 0 = (-a)+a$

(v) Commutativity $a+b = b+a \;\forall\; a, b \in R$

II $(R, \cdot)$ is a semigroup

(i) Closure Property: $a \cdot b \in R \;\forall\; a, b \in R$

(ii) Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c \;\forall\; a, b, c \in R$

(III) Distributive laws:

(i) $a(b+c) = ab + ac$

and (ii) $(a+b)c = ac + bc$

$\forall\, a,b,c \in R$

or $a\cdot(b+c) = a\cdot b + a\cdot c$

or $(a+b)\cdot c = a\cdot c + b\cdot c$.

$\boxed{\begin{array}{l}(R,+,\cdot) \text{ is a ring if} \\ \text{(i) } (R,+) \text{ is abelian group (ii) } (R,\cdot) \text{ is semigroup} \\ \text{(iii) } R \text{ satisfies distributive laws} \\ (a+b)c = ac+bc,\ a(b+c) = ab+ac\end{array}}$

~~Examples~~:

1. $(\mathbb{Z},+,\cdot)$ is a ring.

2. $(\mathbb{Z}_n, +_n, \cdot_n)$ is a ring.

3. $(\mathbb{Q},+,\cdot)$ is a ring

4. $(\mathbb{R},+,\cdot)$ is a ring.

5. $R \to GL(2,\mathbb{R}) \to$ set of all 2×2 matrices over $\mathbb{R}$

operations $\to$ matrix addition $\to (+)$
& matrix multiplication $\Rightarrow (\cdot)$

$(R,+,\cdot)$ is a ring
but $\cdot$ is not commutative (in every case) ($AB \neq BA$)