# Unit-5 : Maxwell Equations & Electromagnetic waves

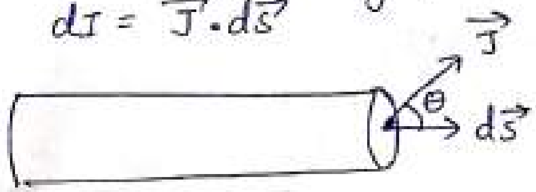
## Equation of continuity

$I = \frac{dq}{dt}$ = Rate of Flow of charge with time (Macroscopic Quantity)

$\vec{J}$ = current Density (Microscopic quantity)
It is characteristic of a point inside the conductor rather than of a conductor as a whole.
Direction of $\vec{J}$ = that of flow of Positive charge.
Magnitude of $|\vec{J}|$ = current per unit area through an infinitesimal area normal to the direction of current flow.
Units = Amp/$m^2$

$J = \frac{I}{S}$ S = area of cross-section

Current through any small area $d\vec{s}$ is given by
$dI = \vec{J} \cdot d\vec{s}$




$\Rightarrow I = \int_S \vec{J} \cdot d\vec{s}$

It depends on the shape of surface taken

$\vec{J}$ is a vector but current $I$ is scalar.
If $\vec{J}$ is const everywhere, current is steady current or ~~steady~~ stationary current

Consider a case in which all charge carriers of charge $q$ move with same velocity $v$.
In time interval $\Delta t$, each charge will travel a distance $v\Delta t = l$
Consider a small volume of cross section ~~ds~~ $ds$ and length $l$.
volume = $d\vec{s} \cdot \vec{l} = d\vec{s} \cdot \vec{v}\Delta t = d\vec{s} \cdot v\, dt$
Let $n$ = no. of charge carriers per unit volume.
then charge in this volume = $q n\, d\vec{s} \cdot \vec{v}\, dt = dq$
$\Rightarrow$ Current = $q n\, d\vec{s} \cdot \vec{v} = \frac{\text{charge}}{dt} = \frac{dq}{dt}$

$$\boxed{\text{Current density} = \frac{I}{ds} = q n \vec{v} = \vec{J}}$$

Note ① If charge carriers have different $n, q$ & $v$

then $I = n_1 q_1 \, d\vec{s} \cdot \vec{v}_1 + n_2 q_2 \, d\vec{s} \cdot \vec{v}_2 + \cdots n_p q_p \, d\vec{s} \cdot \vec{v}_p$

$= d\vec{s} \cdot \sum_{i=1}^{p} n_i q_i v_i$

$\Rightarrow \boxed{J = \sum n_i q_i v_i}$

② In conductor, charge carriers are $e^-$

i.e. $q_1 = q_2 \cdots = q_p = -e$

$J = -e \sum n_i v_i$ ($e^-$ travel with diff velocity)

$$v_{avg} = \frac{\sum n_i v_i}{\sum n_i} = \frac{\sum n_i v_i}{n}$$

$\Rightarrow \boxed{\vec{J} = -en \vec{v}_{avg}}$

$\vec{v}_{avg} \Rightarrow$ direction of flow of $e^-$ & current density is in opposite direction.

## Equation of Continuity

Defn: Electric charge can neither be created nor be destroyed by human and the rate of increase of the total charge inside any arbitrary volume must be equal to the net flow of charge into this volume.

$\therefore \; I = -\int_s \vec{J} \cdot d\vec{s}$ —① → −ve sign shows that $I$ is considered as direction of flow of positive charges.

& $I = \frac{dq}{dt}$ = Rate of flow of charge —②

So ① & ②

$$-\int_s \vec{J} \cdot d\vec{s} = \frac{\partial q}{\partial t} = \frac{\partial}{\partial t} \int_v \rho \, dV \quad , \; \rho = \text{volume charge density}$$

using Gauss divergence theorem

$$-\int_s \vec{J} \cdot d\vec{s} = -\int_v (\vec{\nabla} \cdot \vec{J}) \, dv = \int_v \frac{\partial \rho}{\partial t} \, dV$$

or $\int_V \left(\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right) dv = 0$

$\Rightarrow$ Integrand $= 0$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is Equation of Continuity

If the region does not contain a source/sink of current, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = 0}$ (steady current)

or $\oint_S \vec{J}\cdot d\vec{s} = 0$

$\Rightarrow$ charge cannot accumulate anywhere so the lines of current density are continuous.

For metallic conductors, $\boxed{\vec{J} = \sigma\vec{E}}$

$\sigma =$ conductivity of conductor

$\sigma = \frac{1}{\rho} = \frac{1}{\text{Resistivity}}$

i.e. $\boxed{\vec{E} = \rho\vec{J}}$ units of $\rho$ = Vm/Amp = $\Omega$ m

$\sigma$ & $\rho$ are characteristic of a material. ~~rather~~

# Maxwell's Equations

1. Gauss law in Electrostatics

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \oint_S \vec{E}\cdot\vec{ds} = \frac{Q}{\epsilon_0}$$

2. Gauss Law of Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \oint_S \vec{B}\cdot\vec{ds} = 0$$

Monopoles donot exist

3. Faraday's Law of Electrostatic

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \oint_l \vec{E}\cdot\vec{dl} = -\int_S \left(\frac{\partial\vec{B}}{\partial t}\right)\cdot\vec{ds}$$

4. (Ampere's) ~~Faraday's~~ law for Magnetostatics

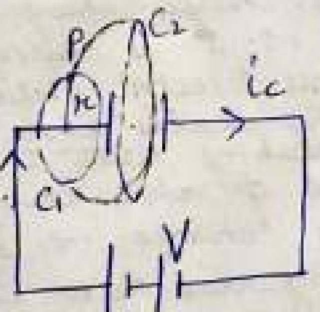
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \oint \vec{B}\cdot\vec{dl} = \mu_0 I_{encl.}$$

$$\vec{\nabla}\cdot (3) \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$$

$$\vec{\nabla}\cdot (4) \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$$

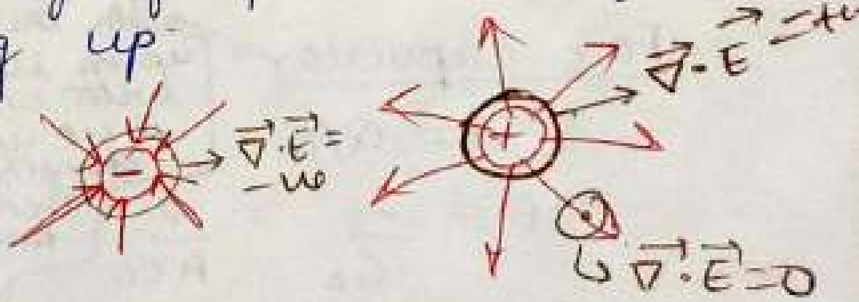
→ not 0 always
0 only in steady state.

$\vec{\nabla}\cdot\vec{J} \neq 0$ → charging of capacitor, charge is piling up




$i_c = i_{enc.}$

eg.




Ampere's circuital law

$$\oint_{C_1} \vec{B}\cdot\vec{dl} = \mu_0 i_{enc}$$

$$B\times 2\pi r = \mu_0 i = \mu_0 i_c$$ → conduction current

But $\oint_{C_2} \vec{B}\cdot\vec{dl} = B\times 2\pi r = 0$ → $\because$ no current exiting
charge is piling up

Hence there is a Contradiction

## Equation of Continuity

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

or $-\vec{\nabla}\cdot\vec{J} = \frac{\partial \rho}{\partial t}$ → something is moving in

Let us consider a volume V having charge density ρ.


Net Rate of Flow of charge into volume = Net Rate of change of charge density (increase)

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla}\cdot\vec{E}\right) = -\epsilon_0\left(\vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\vec{\nabla}\cdot\epsilon_0\frac{\partial \vec{E}}{\partial t} = -\vec{\nabla}\cdot\vec{J}_D$$

or $\vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{J}_D = 0$

So,
$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{B} &= \mu_0\vec{J} + \mu_0\vec{J}_D \\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\end{aligned}}$$ Modified Ampere's Law

$\vec{J}_D = \epsilon_0\frac{\partial \vec{E}}{\partial t}$ Displacement current density

$\vec{\nabla}\cdot\vec{J}_D = I_D$ = Displacement current

**For capacitor**

As the capacitor charges, charge is piling up on its plates i.e. ρ is changing with time. Hence, Electric field E which is set up b/w the plates also changing or increasing with time. Hence flux φ also changes with time. This change i.e. $\frac{\partial E}{\partial t}$ is giving rise to a current in between the plates of capacitor thereby giving rise to Magnetic field. (This is Displacement current) (charges are not moving here)

$$\sigma = Q/A$$

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

$$\frac{\partial E}{\partial t} = \frac{1}{A\epsilon_0}\frac{\partial Q}{\partial t} = \frac{i}{A\epsilon_0}$$

$\vec{\nabla}\times\vec{B} =$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\epsilon_0\int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{s}$$ displacement current

$$= 0 + \mu_0\epsilon_0\frac{i}{A\epsilon_0}\cdot A = \mu_0 i$$

(Non-steady current)

$\vec{J}$ = conduction current, due to flow of charges + $\vec{J}$

$\vec{J_D} \rightarrow$ Displacement current, due to change in Electric field with time = $I_D$

Characteristics of $I_D$

It is equally effective in producing the magnetic field as conduction current. It has no other property of current as it is not linked with motion of charges at all. It makes total current continuous across the discontinuity in conduction current. In good conductors, $I_D$ is negligible as compared to Iconduction for ~~f optical~~ frequency less than $10^{15}$ Hz (optical frequency)

If a material is neither a good conductor nor a dielectric or insulator, then total current density.

$$\vec{J}_{tot} = \vec{J_c} + \vec{J_D}$$

$$= \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}$$

let my field is frequency dependent i.e dependent on $(E_0 e^{j\omega t} = \vec{E})$

$$\vec{J}_{tot} = \sigma\vec{E} + j\omega\epsilon\vec{E}$$

$$\frac{d\vec{E}}{dt} = j\omega E_0 e^{j\omega t} = j\omega\vec{E}$$

$$\left|\frac{J_c}{J_D}\right| = \frac{\sigma}{\omega\epsilon}$$

For pure conductors $J_c$ is very large for frequency < optical frequency

For higher frequency $J_D$ is significant

Note that $I_c = \frac{dq}{dt} = I_D = \epsilon_0\int_s \frac{\partial\vec{E}}{\partial t}\cdot d\vec{s}$

i.e. Conduction Current in leads = Displacement Current

i.e. Continuity is maintained.

# Maxwell Equations (Updated) + Physical Significance
## (In Absence of Dielectric / Magnetic Material) (SI units)

① Gauss's Law for Electric Field

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int_\tau \rho\, d\tau = \frac{Q}{\epsilon_0}$$

or $\vec{\nabla}\cdot\vec{D} = \rho$ ($\vec{D}$ = Electric Displacement vector)

⇒ This gives relation b/w electric field and charges that produce it.
In static situation, it is equivalent to Coulomb's Law
& Relates the electric flux through a closed surface to the charge enclosed.
This law holds for charges in motion also i.e. when Electric field varies with time unlike Coulomb's Law.
It also tells $\vec{E}$ flux lines are continuous.

② Gauss Law for Magnetic Fields

$$\vec{\nabla}\cdot\vec{B} = 0 = \vec{\nabla}\cdot(\mu_0\vec{H}) = \mu_0\vec{\nabla}\cdot\vec{H} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{H} = 0$$

OR

$$\oint_S \vec{B}\cdot d\vec{s} = 0 = \oint_S \vec{H}\cdot d\vec{s}$$

⇒ Unlike electric charges, magnetic charges or magnetic monopoles donot exist i.e.
Magnetic field lines have no sources or sinks to start or stop on.
For eg. Lines of $\vec{B}$ in bar magnet exist inside as well as outside the magnetic material.
It holds even for time dependent magnetic fields. It also tells $\vec{B}$ flux lines are continuous.

## (3) Faraday's Laws of Induction

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{or} \quad \oint_c \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}\cdot d\vec{s} = -\frac{\partial \phi_B}{\partial t} \quad (A)$$

It states the relation b/w induced Electric field generated by a changing magnetic flux. It shows that a varying magnetic field (with time) generates Electric field. The Negative sign is in accordance with Lenz's Law i.e. The induced Electric field will give rise to an induced current that opposes the change in the magnetic flux.

⇒ The $\vec{E}$ produced by a changing magnetic flux is not conservative. This line integral must be carried out over a closed stationary path.

Now $\vec{E} = \vec{E_C} + \vec{E_N}$ (Total)

where $\vec{E_C}$ is field due to static distribution of charges and

$$\oint \vec{E_C}\cdot d\vec{l} = 0 \quad // \text{Conservative} \quad -(B)$$

$\vec{E_N}$ is due to magnetically induced i.e. changing magnetic flux & is Non-Conservative

and $\oint_s \vec{E_N}\cdot d\vec{s} = 0$ (No enclosed static charges in Gaussian surface producing $\vec{E_N}$)

$$\oint_c \vec{E_N}\cdot d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}\cdot d\vec{s} = -\oint_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \quad -(C)$$

Eqn (A) contains information for both (B) & (C). (B) eqn however has 0 contribution on RHS.

## (4) Modified Ampere Circuital Law / Generalized Ampere Law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = \mu_0(\vec{J}+\vec{J}_D)$$

where $\vec{J} = \vec{J}_c$ is conduction current density

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0\oint_s\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{s}$$

$$= \mu_0 I + \mu_0\epsilon_0\oint_s\frac{\partial\vec{E}}{\partial t}\cdot d\vec{s}$$

$$= \mu_0 I + \mu_0\epsilon_0\frac{\partial\phi_E}{\partial t}$$

$\phi_E$ = Electric Flux

It shows that magnetic field is produced by an electric current (I) or by a changing electric field (electric flux).

The second term representing rate of change of electric field flux is known as displacement current distribution

So, 2 possible sources of Magnetic field are Conduction Current & Displacement Current

Q Derive Continuity Eqn from Maxwell's Eqn

→ From 4th Max eqn

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad -(1)$$

Take divergence on both sides

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J}) + \mu_0\epsilon_0\,\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}$$

LHS = 0 ∵ divergence of curl of a vector is 0 (vector identity)

$$\Rightarrow 0 = \mu_0\left(\vec{\nabla}\cdot\vec{J} + \epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})\right)$$

From Gauss Law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\Rightarrow 0 = \mu_0\left(\vec{\nabla}\cdot\vec{J} + \epsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right)\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

Similarly vice versa

Maxwell eqns in Gaussian system of units

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Maxwell Equations in Free space (SI units)

In free space, $\rho = 0$, $\sigma = 0$, $\Rightarrow \vec{J} = \sigma\vec{E} = 0$

(1) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = 0$ (2) $\vec{\nabla}\cdot\vec{B} = 0$

(3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t}$, $\mu_0$ (permeability of free space)

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\mu_0\vec{H}) = 0 + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

(4) $\Rightarrow \vec{\nabla}\times\vec{H} = \epsilon_0\frac{\partial\vec{E}}{\partial t}$, $\epsilon_0$ (permittivity of free space)

For a general Linear Isotropic Medium of permittivity $\epsilon$, permeability $\mu$ & conductivity $\sigma$

$\vec{D} = \epsilon \vec{E}$

$\vec{B} = \mu \vec{H}$

$\vec{J} = \sigma \vec{E}$

$\epsilon_r = 1$ for Isotropic medium i.e. $\epsilon = \epsilon_0$

$\rho$ = Volume charge Density NOT Resistivity

Maxwell eqn

(1) $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ or $\vec{\nabla} \cdot \vec{D} = \rho$

(2) $\vec{\nabla} \cdot \vec{B} = \vec{0}$ or $\vec{\nabla} \cdot \vec{H} = 0$

(3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \left( \frac{\partial \vec{H}}{\partial t} \right)$

(4) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times (\mu \vec{H})$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$

Note: (1) For Dielectric/Non-Conducting medium $\rho = 0$, $\sigma = 0$

(2) Conducting Medium $\rightarrow \rho = 0$, $\sigma \neq 0$

(3) Isotropic Medium $\mu_r > 1$, $\epsilon_r = 1$ (Free space, $\mu = \mu_0$, $\epsilon = \epsilon_0$, $\rho = 0$, $\sigma = 0$)

(4) Anisotropic Medium $\mu_r = 1$ (Non-magnetic) $\epsilon_r$ = Tensor, Properties are directionally dependent

(5) Linear Medium $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E}$

# Electromagnetism / Electromagnetic Waves

→ Maxwell equations tell that a time varying magnetic field acts as a source of Electric field (Inductors/Transformers) & a Time varying Electric field (Displacement current) acts as a source of magnetic field.

Thus, when either $\vec{E}$ or $\vec{B}$ is changing with time, a field of the other kind is induced in adjacent region of space.

→ Thus, we are led to consider the possibility of an electromagnetic Disturbance, consisting of time varying electric & magnetic field, that can propagate through space from one region to another, even when there is no matter in the intervening region. Such a disturbance will have the properties of a wave called as Electromagnetic waves.

eg. Radio wave, TV waves, X rays, light

→ So electricity & magnetism are related via Maxwell eqns. There exist symmetry in ME for free space.

## In Free space

$\sigma = 0$ (No conductivity)

$\rho = 0$ (No volume charge density) (No flow of charges)

$\Rightarrow J_c = 0 \Rightarrow J_c = 0$ (No conduction current)

$\mu = \mu_0, \quad \epsilon = \epsilon_0$ {Free space parameters}

(1) $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\nabla}\cdot\vec{D} = 0 \quad \because \vec{D} = \epsilon_0 \vec{E}$

$$\oint_s \vec{E}\cdot d\vec{s} = \oint_s \vec{D}\cdot d\vec{s} = 0 \quad \text{(surface integral)}$$

(2) $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot\vec{H} = 0 \quad \because \vec{B} = \mu_0\vec{H}$

$$\oint_s \vec{B}\cdot d\vec{s} = \oint_s \vec{H}\cdot d\vec{s} = 0 \quad \text{(~~line~~ surface integral)}$$

(3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$$\oint_c \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_s \vec{B}\cdot d\vec{s}$$

(4) $\vec{\nabla}\times\vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ since $J_c = 0 = \sigma E = 0$

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0 \frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{s} = \oint_c \mu_0 \vec{H}\cdot d\vec{l}$$

## Key Properties of EM waves

(1) Wave is transverse i.e. both $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation of wave. Electric & magnetic fields are also perpendicular to each other. The direction of propagation is the direction of $\vec{E} \times \vec{B}$. $\Rightarrow$ $\vec{E}, \vec{B}$ & $\vec{k}$ are mutually perpendicular or orthogonal

(2) $E = cB$ (where $c$ = vel. of light) in free space

(3) Wave travels in vacuum (free space) with a definite & unchanging speed and velocity of EM wave is less than c when it travels in a medium ($v < c$)

$$n = \frac{c}{v}, \quad n = \text{refractive index}$$

(4) Unlike mechanical waves, which needs the oscillating particles of a medium, EM wave require NO medium. The "waving" in EM wave are the electric & Magnetic field themselves.

(5) EM waves have the property of Polarization

(6) EM waves carry energy & Momentum

(7) In free space $\frac{E}{H} = 377\,\Omega$ = Wave Impedence

$\Rightarrow$ $\vec{E}$ & $\vec{H}$ are in same phase i.e. same relative magnitude at all points at all times

## Classical Wave Equation (Free space)

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2} \quad \text{in 1D} \quad \text{— (A)}$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \vec{\nabla}^2 \psi \quad (3D) \quad \text{— (B)}$$

$v$ = velocity of wave

Let us derive the equations for EM wave by the use of Maxwell's Equations. This is one of the most important applications of ME

## Maxwell's Equation in free space ($\rho = 0, \sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$)

1) $\vec{\nabla} \cdot \vec{D} = 0 = \vec{\nabla} \cdot \vec{E}$

2) $\vec{\nabla} \cdot \vec{B} = 0$

3) $\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

4) $\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\rho = 0$, $\sigma = 0 \Rightarrow \vec{J} = 0$

$\mu = \mu_0$, $\epsilon = \epsilon_0$

Taking curl of $\vec{E}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \left( \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} \right) = -\mu_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

Substituting from 4)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t} \left( \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$= (\vec{\nabla} \cdot \vec{E}) \vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla}) \vec{E}$$

0 (from ①)

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad — ③$$

→ D alembert operator

Comparing it with classical wave equation (B)

$$\vec{\nabla}^2 \vec{E} = \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$v^2 = \frac{1}{\mu_0 \epsilon_0} \quad \text{or} \quad \boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}} \quad 4)$$

Il_ly Analogy to ③

$$\boxed{\vec{\nabla}^2 \vec{H} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}} \quad ⑤$$

(By taking curl of 4th Maxwell equation)

$\mu_0 = 4\pi \times 10^{-7}$ H/m, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m

$$v = \sqrt{\frac{1}{\mu_0 \epsilon_0}} = \sqrt{\frac{1}{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} \text{ m/s}$$

$$\boxed{v = 3 \times 10^8 \text{ m/s} = c = \text{speed of light}} \quad -⑥$$

③ & ⑤ are wave eqns governing electromagnetic fields E and H in free space.

From ⑥, we conclude that EM waves (light wave) are transverse in nature & propagate with velocity of light c.

## Plane Electromagnetic waves in Free space

A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction.

Transverse nature ⇒ field vector is perpendicular to the direction of propagation.

We know that

$\vec{E_0}$, $\vec{H_0}$ >> Amplitude vector (3D)

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$
$$\vec{\nabla}^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0 \quad \text{— (1)}$$

→ 2 wave eqns in Free space where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

The plane wave solⁿ of above equations can be written in the form

$$\vec{E} = \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{— (2)}$$

$$\text{|||}^{\text{ly}} \quad \vec{H} = \vec{H_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{— (3)}$$

(in 3D)

In 1D

$E = E_0\, e^{i(kx - \omega t)}$

OR

$E = E_0 \cos(kx - \omega t)$

$\vec{k}$ = wave propagation vector $= \frac{2\pi}{\lambda}\hat{n}$

$\hat{n}$ is a unit vector in the direction of wave propagation.

According to Maxwell's Equations

1) $\vec{\nabla}\cdot\vec{E} = \vec{0}$

using (2)

iota

$$\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]\cdot\left[E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{— (4)}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{I}^{st} \text{ term} = \frac{\partial}{\partial x}\left[E_{0x}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

$$= E_{0x} \frac{\partial}{\partial x}\left(e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$$

$$= E_{0x}(ik_x)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$$
$$\vec{E} = E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}$$

$\therefore$ (4) $\Rightarrow$

$$\vec{\nabla}\cdot\vec{E} = \left[E_{0x}(ik_x) + E_{0y}(ik_y) + E_{0z}(ik_z)\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= i(\vec{E_0}\cdot\vec{k})\, \underbrace{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}_{=\vec{E}} = 0$$

$$\Rightarrow i(\vec{E}\cdot\vec{k}) = 0$$

$$\Rightarrow \vec{k}\cdot\vec{E} = 0$$

$$\Rightarrow \boxed{\vec{k} \perp \vec{E}}$$

Similarly $\longrightarrow$ Do it yourself

1) $\vec{\nabla}\cdot\vec{H} = 0$

$$\Rightarrow i(\vec{H_0}\cdot\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$

$$\Rightarrow i(\vec{H}\cdot\vec{k}) = 0$$

$$\Rightarrow \vec{H}\cdot\vec{k} = 0$$

$$\Rightarrow \boxed{\vec{k} \perp \vec{H}}$$

So we conclude that EM waves are transverse in nature as $\vec{E}$ and $\vec{H}$ are both perpendicular to direction of propagation vector $\vec{k}$.

Using 3) Maxwells Equation

3) $\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$\Rightarrow \vec{\nabla} \times \left[\vec{E_0} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right] = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

RHS $= -\mu_0 \frac{\partial}{\partial t}\left[H_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$

$= -\mu_0 (-j\omega) H_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\Rightarrow j\omega\mu_0 \vec{H}$

LHS $\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)} & E_{0y} e^{i(\vec{k}\cdot\vec{r} - \omega t)} & E_{0z} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{vmatrix}$

$= \Big\{ \hat{i}\left[E_{0z}(jk_y) e - E_{0y}(jk_z) e\right]$
$- \hat{j}\left[E_{0z}(jk_x) - E_{0x}(jk_z)\right]$
$+ \hat{k}\left[E_{0y}(jk_x) - E_{0x}(jk_y)\right]\Big\} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$= j\Big\{(E_{0z}k_y - E_{0y}k_z)\hat{i} - (E_{0z}k_x - E_{0x}k_z)\hat{j}$
$+ (E_{0y}k_x - E_{0x}k_y)\hat{k}\Big\} e^{j(\vec{k}\cdot\vec{r} - \omega t)}$

$= j(\vec{k} \times \vec{E}) = j\omega\mu_0 \vec{H}$

$\Rightarrow \boxed{\vec{k} \times \vec{E} = \omega\mu_0 \vec{H}}$ — (8)

4) ~~Mr~~ using $\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ (Do it yourself)

we get $\boxed{\vec{k} \times \vec{H} = -\omega\epsilon_0 \vec{E}}$ — (9)

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We infer from (8) & (9) that
$\vec{H}$ is perpendicular to plane formed by $\vec{k}$ & $\vec{E}$
$\vec{E}$ is perpendicular to plane formed by $\vec{k}$ & $\vec{H}$
$\Rightarrow$ $\vec{H}, \vec{k}$ and $\vec{E}$ are mutually perpendicular to each other and even perpendicular to direction of EM wave i.e. $(\vec{E}, \vec{H}, \vec{k})$ They form ortho-normal set (orthogonal set of vectors)

$$\vec{H} = \frac{1}{\omega\mu_0}(\vec{k}\times\vec{E}) = \frac{k}{\omega\mu_0}(\hat{n}\times\vec{E}) \quad \because \vec{k} = k\hat{n}$$

$$H = \frac{kE}{\omega\mu_0} \quad \text{modulus form (magnitude)}$$

$$H = \frac{\cancel{\omega}}{\cancel{\omega}c\mu_0}E \qquad \because c = \frac{\omega}{k} = \text{vel. of em wave in free space}$$

$$\frac{\mu_0}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

$$\therefore \frac{E}{H} = \mu_0 c = \frac{\mu_0}{\sqrt{\mu_0\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\frac{E}{H} = \sqrt{\frac{4\pi\times10^{-7}\ H/m}{8.85\times10^{-12}\ F/m}} = 376.82\ \Omega \simeq 377\ \Omega = Z$$

$$\boxed{Z \simeq 377\ \Omega}$$ wave impedance of free space

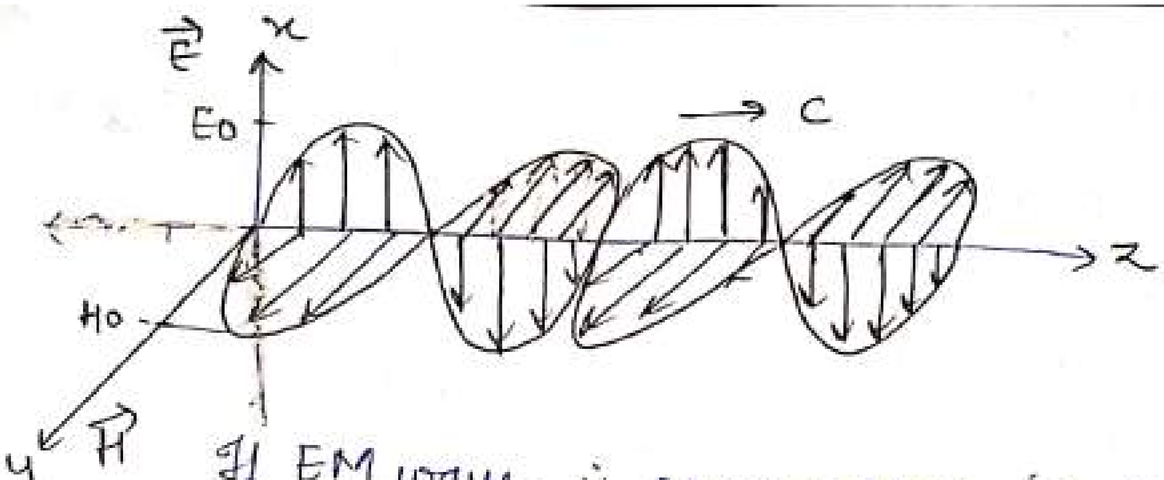
$\because$ $E \to V/m$ $\quad H \to$ Amp-turn/m

$V/A = \Omega$

$Z = \left|\frac{E}{H}\right|$ is real and positive

$\Rightarrow$ Field vectors $\vec{E}$ and $\vec{H}$ are in the same phase i.e. they have the same relative magnitude at all points at all times

$$\boxed{E = c\mu_0 H = cB}$$




If EM wave is propagating in medium

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_r\epsilon_0}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} = \frac{c}{n} \quad -(10)$$

where $\mu_r, \epsilon_r$ = Relative permeability & Relative permittivity of medium (+ve constt)

$n = \sqrt{\mu_r\epsilon_r}$ = Refractive index of medium

For non-magnetic material

$\mu_r = 1$

$\therefore \boxed{n = \sqrt{\epsilon_r}}$ = Maxwell's Relation

(10) $\Rightarrow$ that the speed of EM waves in an dielectric (isotropic) is less than the speed of EM waves in free space

## Energy ~~and M.~~ in EM waves

Energy density associated with $\vec{E} = \frac{1}{2}\epsilon_0 E^2$

Energy density associated with $\vec{B} = \frac{1}{2\mu_0} B^2$

Total energy Density in EM wave

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0} \quad -(1)$$

In free space, $B = \frac{E}{c} = \sqrt{\mu_0 \epsilon_0}\, E \quad -(2)$

in (1)

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\mu_0 \epsilon_0 E^2$$

$$\boxed{U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2}$$ Energy Density $J/m^3$

⇒ In vacuum/free space, energy density of $\vec{B} = \frac{1}{2}\epsilon_0 E^2$ is equal to energy density of $\vec{E}$

In general, $\vec{E}, \vec{B}$ is function of position & time

⇒ U is also dependent on position & time

⇒ EM waves (travelling) transport Energy from one region to another

Energy $dE = U\, dV$     $dV$ = Volume

$dE = (\epsilon_0 E^2)(A\, c\, dt)$     → Area of stationary plane = A
→ $c\,dt$ is the distance travelled by wavefront

$S = \frac{1}{A}\frac{dE}{dt}$ = Energy Flow per unit time per unit area = Power per unit area for an area perpendicular to the direction of wave travel

⇒ $\boxed{S = \epsilon_0 c E^2}$ in vacuum $-(3)$

⇒ $\boxed{S = \frac{\epsilon_0}{\sqrt{\mu_0 \epsilon_0}} E^2 = \sqrt{\frac{\epsilon_0}{\mu_0}}\, E^2 = \frac{EB}{\mu_0}}$ $\because c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ $-(4)$

SI unit of $S = J/(s\, m^2) = Watt/m^2$

Defining S in terms of a vector
From (9)

$$\boxed{\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}}$$ Poynting vector in vacuum (Intensity of Radiation)

Total energy flow per unit time (or Power P) out of any closed surface is

$$P = \oint \vec{S} \cdot d\vec{A}$$ area

# $S_{avg} = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2$

$$\boxed{S_{avg} = \frac{1}{2} \epsilon_0 c E_0^2}$$ Average value

# Momentum Density $= \frac{dp}{dV} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2}$

$dV$ = volume
(This momentum is not associated with mass)